

Explicit evolution relations with orbital elements for eccentric, inclined, elliptic and hyperbolic restricted few-body problems

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Abstract Planetary, stellar and galactic physics often rely on the general restricted gravitational N -body problem to model the motion of a small-mass object under the influence of much more massive objects. Here, I formulate the general restricted problem entirely and specifically in terms of the commonly-used orbital elements of semimajor axis, eccentricity, inclination, longitude of ascending node, argument of pericentre, and true anomaly, without any assumptions about their magnitudes. I derive the equations of motion in the general, unaveraged case, as well as specific cases, with respect to both a bodycentric and barycentric origin. I then reduce the equations to three-body systems, and present compact singly- and doubly-averaged expressions which can be readily applied to systems of interest. This method recovers classic Lidov-Kozai and Laplace-Lagrange theory in the test particle limit to any order, but with fewer assumptions, and reveals a complete analytic solution for the averaged planetary pericentre precession in coplanar circular circumbinary systems to at least the first three nonzero orders in semimajor axis ratio. Finally, I show how the unaveraged equations may be used to express resonant angle evolution in an explicit manner that is not subject to expansions of eccentricity and inclination about small nor any other values.

1 Overview

The movement of an infinitesimal mass in a region dominated by massive bodies has important implications for designing spacecraft missions (Gómez et al., 2001), preparing for near-Earth interlopers (Shoemaker 1995; de la Fuente Marcos & de la Fuente Marcos 2013), and understanding the behaviour of

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planetary, stellar and galactic systems (Binney & Tremaine 1987; Murray & Dermott 1999). Applications are far-reaching (e.g. to black holes, Schnittman, 2010). This resulting motion is related, but not strictly equivalent, to the motion found in the *restricted problem*.

1.1 Context

The seminal work of Szebehely (1967) claims to be “the first book devoted to the theory of orbits in the restricted problem”. His historical perspective highlights the inherent assumptions which accompany the term *restricted* and have since been reinforced by later celestial mechanics texts (pg. 253 of Danby 1992, pg. 63 of Murray & Dermott 1999). These assumptions are i) the system contains three bodies, two of which are massive, ii) the two massive bodies share a mutual circular orbit, and iii) all three bodies have coplanar orbits. Other texts have begun to explicitly use the terms *circular* or *planar* to qualify the otherwise broad terminology (pg. 196 of Morbidelli 2002, pg. 118 of Roy 2005, pg. 115 of Valtonen & Karttunen 2006)

Standard treatments of this famous but quite specific case follow a similar pattern of deriving the Cartesian equations of motion by introducing a rotating coordinate system and defining a potential from which zero-velocity surfaces, the Jacobi constant (or integral of motion), the Tisserand parameter and the five Lagrangian equilibrium points may be obtained. Although the resulting equations of motion provide insight into several concepts, such as the Hill sphere, they do not immediately shed light on some basic orbit characteristics such as how the pericentre of the zero-mass body changes with time.

1.2 Objective

In this paper, I derive the equations of motion for the *general* restricted problem in terms of solely the semimajor axis, a , eccentricity, e , inclination, i , longitude of ascending node, Ω , argument of pericentre, ω , and true anomaly, f of the zero-mass body and of all of the massive bodies. I will also use the mean motion n as a convenient auxiliary parameter than can be expressed solely in terms of a and the masses. The word *general* refers to the removal of all the aforementioned assumptions; the systems here may host an arbitrary number of massive bodies on arbitrary but known orbits. By no means, however, is this case the most general type of restricted problem (see Chapter 1.9 of Szebehely 1967 for other extensions).

Importantly, I present the unaveraged equations as well as the averaged equations; the latter case for the unrestricted 3-body problem has been scrutinized in depth-recently, largely since the initial discovery and confirmation of extrasolar planets (Wolszczan & Frail, 1992; Wolszczan, 1994). The general restricted equations I present here are not confined to small mass ratios (as is characteristic of the related work of Henri Poincaré), small perturbative

forces, nor any type of expansion about a limiting orbital element value. The resulting relations may be potentially useful tools which can be applied to a problem of interest. Such problems need not contain a test particle; as long as the smallest mass is much smaller than the other masses, the equations will describe the motion to a good approximation.

1.3 Benefits of orbital element approach

Small-body, planetary and stellar dynamicists often rely on the set of elements $(a, e, i, \Omega, \omega)$ to obtain an intuitive feel for the osculating motion. The location of an object along its orbit can be gleaned from f , or alternatively the mean anomaly, mean longitude or true longitude. All these elements directly demonstrate, for example, how close or how far an object may extend from a massive body, and are easily amenable to limiting cases. For example, classic Lidov-Kozai theory, which assumes the presence of a test particle, is based on the interplay between e and i . Also, one reason why the Tisserand parameter is so useful is because it relates a , e and i to one another.

Observational data is another major motivation for using orbital elements. The majority of extrasolar planets have been discovered by Doppler radial velocity spectroscopy, which yields an observable from which e , ω and f could be measured with a fit to the data. Further, the three major exoplanet databases (see the Extrasolar Planets Encyclopedia at <http://exoplanet.eu/>, the Exoplanet Data Explorer at <http://exoplanets.org/> and the NASA Exoplanet Archive at <http://exoplanetarchive.ipac.caltech.edu/>) all report data in terms of orbital elements. Finally, for purposes of direct integration of a known stellar or planetary system, avoiding scaled Cartesian coordinates removes the need to convert both the input and output.

1.4 How to use this paper

The reader can use the equations in this paper i) for direct integration to solve for the motion of the zero-mass body¹, ii) to obtain physical intuition for what orbital properties are the most significant catalysts of orbital variation, iii) to treat a wide variety of restricted problems in a consistent analytical framework, and iv) to derive existing theories in an alternate manner. The only key assumptions made throughout the paper is that the object I classify as the secondary contains no mass and an osculating elliptical orbit, and that the orbits of all other bodies are known functions of time.

The reader should first identify the number of bodies in their restricted problem, assumptions about their orbits, and the reference point from which to measure orbital elements. Then scanning Fig. 1 will help identify the appropriate setup. Each red line in the figure refers to the orbital plane of the

¹ No integration is necessary for equation (212).

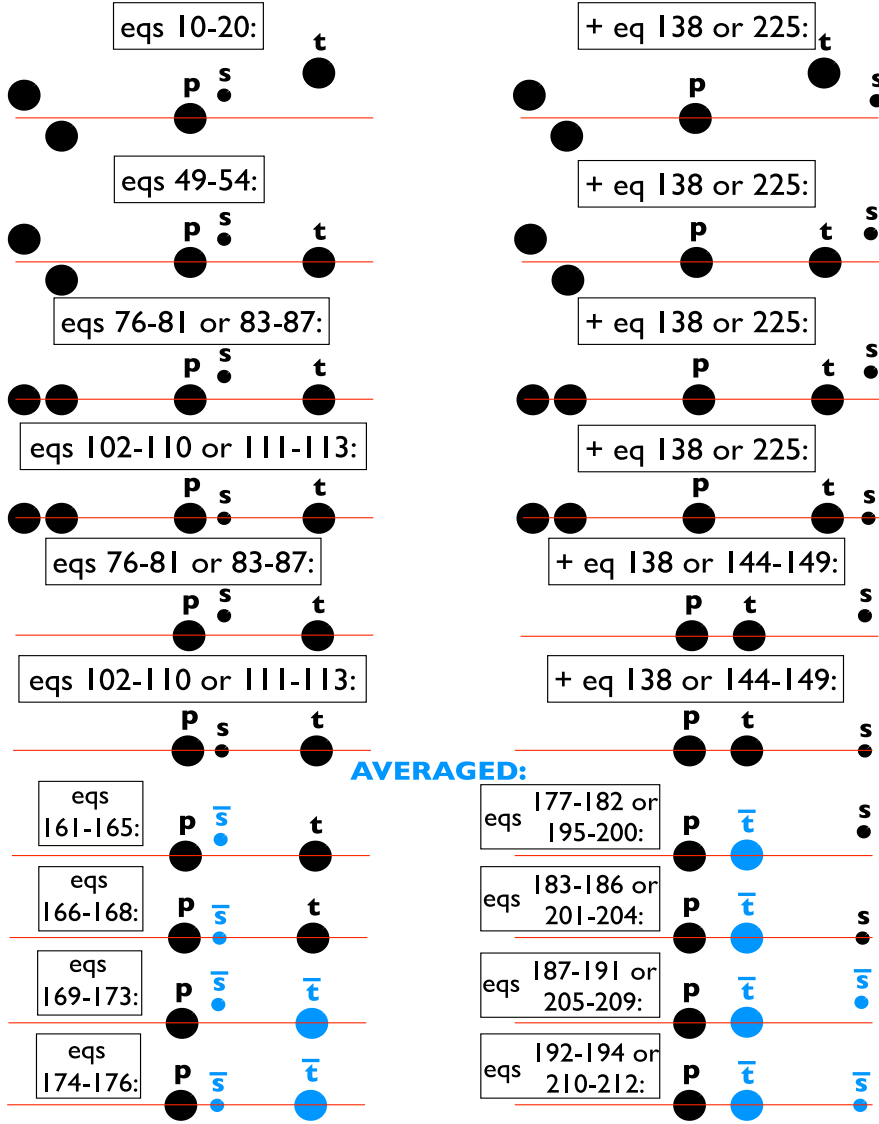


Fig. 1 Representative cartoons of different restricted N -body problems considered in this paper, along with the corresponding equation numbers describing the equations of motion. Here “p”, “s” and “t” refer to the primary, secondary and tertiary; the secondary is always massless and all other bodies are always massive. Although a fourth and fifth body are present in the top 8 configurations, these bodies are merely a proxy for an arbitrary number of bodies. Each red line is the orbital plane of the primary; other bodies placed on that line share elliptic or hyperbolic coplanar orbits with the primary. The left column refers to setups where the orbital elements are measured with respect to the primary (typically when the other massive bodies are exterior to the secondary), and the right column where the orbital elements are measured with respect to the barycentre of a given number of massive bodies (typically when the secondary’s orbit is exterior to more than one massive body). A blue body containing an overbar indicates that body’s orbit is averaged over its true anomaly. Averaged tertiary orbits are assumed to be elliptical.

massive primary “p”. The left column features setups where the orbital elements of the massless secondary “s” is measured with respect to the primary; in the right column the secondary’s orbital elements are measured with respect to the barycentre of more than one of the massive bodies. The top eight configurations generally refer to the N -body problem (not specifically the 5-body problem), and the bottom eight configurations all showcase averaged elliptical orbits.

1.5 Outline of paper

That figure provides specific equation numbers, but here I describe the content of the various sections. First, I set up the problem in Section 2 before describing the derivation technique in Section 3. The next three sections (4-6) present the equations of motion for, first, an arbitrary number of bodies on arbitrary orbits, then when one reference plane of one or more of the massive bodies is fixed, and finally for the assumption that all bodies have forever coplanar orbits. These equations all assume that the orbital elements are measured with respect to the primary. Section 7 briefly touches on what modifications to the equations can be made when only three bodies are in the system.

The paper then transitions, and evaluates how the equations would be transformed if orbital elements were measured with respect to some barycentric reference frame. Section 8 presents the three-body case, and Appendix A presents the general case. Section 8 contains both the necessary scaling form and the explicit equations themselves.

Up until that point, all equations considered will have been unaveraged, and contain the true anomalies of all of the bodies in the system. Sections 9-12 consider averaged cases for the three-body problem. I consider every type of averaging for an internal (Section 10) and external (Sections 11-12) secondary. Section 12 considers the relevant and analytically tractable case of a primary-tertiary pair on a circular orbit. A brief exposition on resonances follows in Section 13, and Section 14 summarizes this work.

2 Setup

Consider a system that contains $N \geq 3$ gravitationally interacting point masses m_j , where gravity is the only acting force, and $j = 3 \dots N$. Assume the position of the secondary with respect to the primary is denoted by $\mathbf{r} = (x, y, z)$ and the position of all other bodies with respect to the primary by $\mathbf{r}_j = (x_j, y_j, z_j)$. The massive primary ($m_1 \equiv m_p$) and massless secondary ($m_2 \equiv m_s = 0$) are assumed to be initially bound to one another. The tertiary mass is denoted by $m_3 \equiv m_t$. In effect, the equations of motion can be applied for a relatively small but nonzero m_s to an excellent approximation. An example of one configuration is a Solar-type star (primary), an asteroid (secondary), a terrestrial planet (tertiary) and a Jovian planet (quaternary), where the motions of the tertiary and quaternary about the primary are known.

3 Derivation technique

The general restricted system contains no known constants of the motion. Neither energy nor angular momentum is conserved. The Jacobi constant and the Tisserand parameter do not apply, except in a specific case. Without these tools to help derive the equations, I instead turn to perturbation theory, where the perturbation may be arbitrary large.

Lagrange’s planetary equations are useful here because they are derived without approximation (e.g. Brouwer & Clemence, 1961). Other derivations, such as for evolution equations described by given radial, tangential and normal components of a perturbative force (Burns 1976 and pgs. 54-57 of Murray & Dermott 1999) are not used because they assume that the perturbed force is small. Lagrange’s planetary equations traditionally contain a truncated disturbing function, but need not. The equations can instead be expressed as equation (22) of Efroimsky (2005) and equation (16) of Gurfil (2007), for an arbitrary perturbative acceleration and in terms of precomputed matrices of Poisson Brackets and partial derivatives of positions with respect to orbital elements². The relevant equations can be found in Veras & Evans (2013) and are not repeated here.

The form of the perturbative acceleration is the key to application of the method. The acceleration must be a function of the position and velocity of the secondary only, and must be a simple enough function of the positions and velocities to be analytically tractable. Denote this acceleration as Δ . Then

$$\underbrace{\frac{d^2 \mathbf{r}}{dt^2} = -\frac{G(m_p + m_s) \mathbf{r}}{r^3}}_{\text{classic 2-body problem}} + \underbrace{\Delta}_{\text{perturbation}} \quad (1)$$

where the arbitrarily large perturbative accelerations on the secondary orbit are $\Delta = \sum_{j=3}^N (\Delta_{j,A} + \Delta_{j,B})$ where

$$\Delta_{j,A} = -\frac{Gm_j \mathbf{r}_j}{r_j^3} \quad (2)$$

and

$$\Delta_{j,B} = \frac{Gm_j (\mathbf{r}_j - \mathbf{r})}{|\mathbf{r}_j - \mathbf{r}|^3}. \quad (3)$$

Although Δ contains \mathbf{r}_j terms, they are, crucially, independent of the secondary’s position and velocity because the secondary has zero mass. Further, I find that the functional dependence on (x, y, z) is not complex enough to prevent the method from succeeding. The time evolution of the secondary’s orbital elements is additive so that they can be decomposed into separate terms attributable to both $\Delta_{j,A}$ and $\Delta_{j,B}$. I obtain

² The derivation of Lagrange’s planetary equations contains a previously missed degree of freedom (Efroimsky & Goldreich, 2003, 2004), which, although not exploited here, may be applied in future studies to obtain new insight into the motion.

$$\frac{da}{dt} = \sum_{j=3}^N \left[\left(\frac{da}{dt} \right)_{j,A} + \left(\frac{da}{dt} \right)_{j,B} \right], \quad (4)$$

$$\frac{de}{dt} = \sum_{j=3}^N \left[\left(\frac{de}{dt} \right)_{j,A} + \left(\frac{de}{dt} \right)_{j,B} \right], \quad (5)$$

$$\frac{di}{dt} = \sum_{j=3}^N \left[\left(\frac{di}{dt} \right)_{j,A} + \left(\frac{di}{dt} \right)_{j,B} \right], \quad (6)$$

$$\frac{d\Omega}{dt} = \sum_{j=3}^N \left[\left(\frac{d\Omega}{dt} \right)_{j,A} + \left(\frac{d\Omega}{dt} \right)_{j,B} \right], \quad (7)$$

$$\frac{d\omega}{dt} = \sum_{j=3}^N \left[\left(\frac{d\omega}{dt} \right)_{j,A} + \left(\frac{d\omega}{dt} \right)_{j,B} \right], \quad (8)$$

$$\frac{df}{dt} = \left(\frac{df}{dt} \right)_{\text{unperturbed 2-body}} + \sum_{j=3}^N \left[\left(\frac{df}{dt} \right)_{j,A} + \left(\frac{df}{dt} \right)_{j,B} \right]. \quad (9)$$

The unperturbed two-body term describes the orbital evolution of the classic two-body problem. In order to derive the desired equations from Δ , I follow the same algebraic procedure described in Veras & Evans (2013). Now I begin presenting the results.

4 General equations in the inertial frame

In the general restricted N -body problem, the equations of motion for the massless secondary's orbit are

$$\begin{aligned} \left(\frac{da}{dt} \right)_{j,A} &= \frac{2Gm_j}{n\sqrt{1-e^2}r_j^3} [x_j (C_1 \cos i \sin \Omega + C_2 \cos \Omega) \\ &\quad + y_j (-C_1 \cos i \cos \Omega + C_2 \sin \Omega) - z_j (C_1 \sin i)], \end{aligned} \quad (10)$$

$$\begin{aligned} \left(\frac{de}{dt} \right)_{j,A} &= \frac{Gm_j\sqrt{1-e^2}}{2an(1+e\cos f)r_j^3} [x_j (C_6 \cos i \sin \Omega + C_5 \cos \Omega) \\ &\quad + y_j (-C_6 \cos i \cos \Omega + C_5 \sin \Omega) - z_j (C_6 \sin i)], \end{aligned} \quad (11)$$

$$\begin{aligned} \left(\frac{di}{dt} \right)_{j,A} &= -\frac{Gm_j\sqrt{1-e^2}}{an(1+e\cos f)r_j^3} \cos(f+\omega) [x_j (\sin i \sin \Omega) \\ &\quad - y_j (\sin i \cos \Omega) + z_j (\cos i)], \end{aligned} \quad (12)$$

$$\left(\frac{d\Omega}{dt} \right)_{j,A} = -\frac{Gm_j\sqrt{1-e^2}}{an(1+e\cos f)r_j^3} \sin(f+\omega) [x_j (\sin \Omega)$$

$$- y_j (\cos \Omega) + z_j (\cot i)], \quad (13)$$

$$\begin{aligned} \left(\frac{d\omega}{dt} \right)_{j,A} &= \frac{Gm_j \sqrt{1-e^2}}{2aen(1+e\cos f)r_j^3} [x_j (-C_8 \cos i \sin \Omega + C_7 \cos \Omega) \\ &\quad + y_j (C_8 \cos i \cos \Omega + C_7 \sin \Omega) \\ &\quad + z_j (C_9 \sin i + 2e \sin(f+\omega) \cos i \cot i)] \end{aligned} \quad (14)$$

and

$$\begin{aligned} \left(\frac{da}{dt} \right)_{j,B} &= \frac{2Gm_j}{n\sqrt{1-e^2}r_{j,B}^3} [x_{j,B} (-C_1 \cos i \sin \Omega - C_2 \cos \Omega) \\ &\quad + y_{j,B} (C_1 \cos i \cos \Omega - C_2 \sin \Omega) + z_{j,B} (C_1 \sin i)], \end{aligned} \quad (15)$$

$$\begin{aligned} \left(\frac{de}{dt} \right)_{j,B} &= \frac{Gm_j \sqrt{1-e^2}}{2an(1+e\cos f)r_{j,B}^3} [-2a \sin f (1-e^2) \\ &\quad + x_j (-C_6 \cos i \sin \Omega - C_5 \cos \Omega) \\ &\quad + y_j (C_6 \cos i \cos \Omega - C_5 \sin \Omega) + z_j (C_6 \sin i)], \end{aligned} \quad (16)$$

$$\begin{aligned} \left(\frac{di}{dt} \right)_{j,B} &= \frac{Gm_j \sqrt{1-e^2}}{an(1+e\cos f)r_{j,B}^3} \cos(f+\omega) \\ &\quad \times [x_j (\sin i \sin \Omega) - y_j (\sin i \cos \Omega) + z_j (\cos i)], \end{aligned} \quad (17)$$

$$\begin{aligned} \left(\frac{d\Omega}{dt} \right)_{j,B} &= \frac{Gm_j \sqrt{1-e^2}}{an(1+e\cos f)r_{j,B}^3} \sin(f+\omega) \\ &\quad \times [x_j (\sin \Omega) - y_j (\cos \Omega) + z_j (\cot i)], \end{aligned} \quad (18)$$

$$\begin{aligned} \left(\frac{d\omega}{dt} \right)_{j,B} &= \frac{Gm_j \sqrt{1-e^2}}{2aen(1+e\cos f)r_{j,B}^3} \\ &\quad \times \left[-2e \cos i \sin(f+\omega) [x_j (\sin \Omega) - y_j (\cos \Omega) + z_j (\cot i)] \right. \\ &\quad + x_{j,B} (C_9 \cos i \sin \Omega - C_7 \cos \Omega) \\ &\quad \left. + y_{j,B} (-C_9 \cos i \cos \Omega - C_7 \sin \Omega) + z_{j,B} (-C_9 \sin i) \right] \end{aligned} \quad (19)$$

with

$$\frac{df}{dt} = \frac{n(1+e\cos f)^2}{(1-e^2)^{3/2}} - \frac{d\omega}{dt} - \cos i \frac{d\Omega}{dt}. \quad (20)$$

The auxiliary set of C variables depend only on the orbital parameters of the primary-secondary orbit and can be expressed as

$$C_1 \equiv e \cos \omega + \cos(f+\omega), \quad (21)$$

$$C_2 \equiv e \sin \omega + \sin(f+\omega), \quad (22)$$

$$C_3 \equiv \cos i \sin \Omega \sin(f+\omega) - \cos \Omega \cos(f+\omega), \quad (23)$$

$$C_4 \equiv \cos i \cos \Omega \sin (f + \omega) + \sin \Omega \cos (f + \omega), \quad (24)$$

$$C_5 \equiv (3 + 4e \cos f + \cos 2f) \sin \omega + 2(e + \cos f) \cos \omega \sin f, \quad (25)$$

$$C_6 \equiv (3 + 4e \cos f + \cos 2f) \cos \omega - 2(e + \cos f) \sin \omega \sin f, \quad (26)$$

$$C_7 \equiv (3 + 2e \cos f - \cos 2f) \cos \omega + \sin \omega \sin 2f, \quad (27)$$

$$C_8 \equiv (3 - \cos 2f) \sin \omega - 2(e + \cos f) \cos \omega \sin f, \quad (28)$$

$$C_9 \equiv (3 + 2e \cos f - \cos 2f) \sin \omega - \cos \omega \sin 2f. \quad (29)$$

The Cartesian components of the position vectors of all of the massive bodies in orbital elements are

$$x_j = r_j [\cos \Omega_j \cos (f_j + \omega_j) - \sin \Omega_j \sin (f_j + \omega_j) \cos i_j], \quad (30)$$

$$y_j = r_j [\sin \Omega_j \cos (f_j + \omega_j) + \cos \Omega_j \sin (f_j + \omega_j) \cos i_j], \quad (31)$$

$$z_j = r_j [\sin (f_j + \omega_j) \sin i_j] \quad (32)$$

with

$$r_j = \frac{p_j}{1 + e_j \cos f_j} \quad (33)$$

where for elliptical and hyperbolic orbits, $p_j = a_j (1 - e_j^2)$, and $p_j = a_j (e_j^2 - 1)$, respectively. For a parabolic tertiary orbit, p_j equals twice the pericentric distance. The difference between an elliptic and hyperbolic restricted problem resides simply in the definition of r_j in equation (33). Also, $\mathbf{r}_{j,B} \equiv (x_{j,B}, y_{j,B}, z_{j,B})$ with

$$x_{j,B} = x_j + \frac{aC_3 (1 - e^2)}{1 + e \cos f} = x_j + rC_3, \quad (34)$$

$$y_{j,B} = y_j - \frac{aC_4 (1 - e^2)}{1 + e \cos f} = y_j - rC_4, \quad (35)$$

$$z_{j,B} = z_j - \frac{a (1 - e^2) \sin i \sin (f + \omega)}{1 + e \cos f} = z_j - r \sin i \sin (f + \omega). \quad (36)$$

Now the equations of motion have been expressed entirely in terms of orbital elements. I use the definitions of the C variables in order to maintain consistency with Veras & Evans (2013). The form of equations (30)-(32) makes no assumptions about the boundedness of the orbit for the tertiary, as the true anomaly f_j can be defined for all orbit types as the angle between the pericentre and the tertiary's location. Usually, for parabolic and hyperbolic orbits, the reference direction is coplanar with the orbit and coincides with the line between the pericentre and the primary. Hence, in those contexts, the angles ω_j and ϖ_j are rarely used.

5 General equations in the rotated frame

5.1 A fixed primary-tertiary reference plane

I can simplify the equations of motion by tilting the reference frame so that it coincides with the plane of the two-body orbit between the primary and one of the m_j , $j \geq 3$ bodies. Here I use the primary-tertiary orbital plane as the reference plane, with an arbitrary but fixed reference direction within that plane to measure the orbital angles.

This transformation, however, comes at a cost. In order for the equations to be most useful, I must assume that the primary-tertiary orbit plane remains fixed in space and does not precess due to the influence of the bodies denoted by m_j , $j \geq 4$. In reality, the plane will precess and the reference direction will change by some nonzero amount because the $j \geq 4$ bodies are not massless. However, the precession is often negligible in several realistic cases, such as the ecliptic of the Solar System, and four and five-body problems which are hierarchical in mass (for example, a restricted three-body problem contained within a restricted four-body problem)³. Therefore, although the equations in this section for systems with $N \geq 4$ bodies are technically inexact, they may prove useful.

Consequently, the orbital parameters of all other bodies are now measured with respect to this (assumed-fixed) orbital plane and reference direction. If viewed face-on and if the primary and tertiary orbit each other, then the orbital motion of the primary and tertiary can be in one of two directions. As viewed from the north poles of those objects, assume that they orbit in a counterclockwise fashion. Then I can set $i_t = 0^\circ$ and $\varpi_t = \omega_t + \Omega_t$, where ϖ represents the longitude of pericentre⁴. This action allows me to eliminate i_t , ω_t and Ω_t from the equations such that now

$$x_t = r_t \cos(f_t + \varpi_t), \quad (37)$$

$$y_t = r_t \sin(f_t + \varpi_t), \quad (38)$$

$$z_t = 0. \quad (39)$$

and

$$r_{3,B} \equiv r_{t,B} = r_t \left\{ 1 - \kappa_t \left(\frac{r}{r_t} \right) + \left(\frac{r}{r_t} \right)^2 \right\}^{1/2} \quad (40)$$

$$= r \left\{ 1 - \kappa_t \left(\frac{r_t}{r} \right) + \left(\frac{r_t}{r} \right)^2 \right\}^{1/2}. \quad (41)$$

³ The quality of the approximation may be estimated by considering the precession rate of the primary-tertiary orbital plane in the solution of the full three-body problem with the primary, tertiary and the most massive body m_j , $j \geq 4$.

⁴ Alternatively, for clockwise motion I can set $i_t = 180^\circ$ and define an *obverse of pericentre* as in Veras & Evans (2013).

I write $r_{t,B}$ in both these forms to foreshadow future expansions of this variable when averaging the equations of motion. In either case,

$$\begin{aligned}\kappa_t &= 2 [\cos(f + \omega) \cos \chi_t - \cos i \sin(f + \omega) \sin \chi_t] \\ &= 2 (D_{t,2} \cos f - D_{t,1} \sin f)\end{aligned}\quad (42)$$

$$= 2 (D_{t,3} \cos f_t + D_{t,4} \sin f_t) \quad (43)$$

with

$$\chi_t = \Omega - f_t - \varpi_t, \quad (44)$$

$$D_{t,1} = \sin \omega \cos \chi_t + \cos \omega \cos i \sin \chi_t, \quad (45)$$

$$D_{t,2} = \cos \omega \cos \chi_t - \sin \omega \cos i \sin \chi_t, \quad (46)$$

$$D_{t,3} = \cos(f + \omega) \cos(\Omega - \varpi_t) - \cos i \sin(f + \omega) \sin(\Omega - \varpi_t), \quad (47)$$

$$D_{t,4} = \cos(f + \omega) \sin(\Omega - \varpi_t) + \cos i \sin(f + \omega) \cos(\Omega - \varpi_t). \quad (48)$$

Now I can re-express the equations of motion as

$$\begin{aligned}\frac{da}{dt} &= \frac{2Gm_t r_t}{n\sqrt{1-e^2}} \left(\frac{S_{(t,A,a)}}{r_t^3} + \frac{S_{(t,B,a)}}{r_{t,B}^3} \right) \\ &\quad + \sum_{j=4}^N \left[\left(\frac{da}{dt} \right)_{j,A} + \left(\frac{da}{dt} \right)_{j,B} \right],\end{aligned}\quad (49)$$

$$\begin{aligned}\frac{de}{dt} &= \frac{Gm_t \sqrt{1-e^2} r_t}{2an(1+e \cos f)} \left(\frac{S_{(t,A,e)}}{r_t^3} + \frac{S_{(t,B,e)}}{r_{t,B}^3} \right) \\ &\quad + \sum_{j=4}^N \left[\left(\frac{de}{dt} \right)_{j,A} + \left(\frac{de}{dt} \right)_{j,B} \right],\end{aligned}\quad (50)$$

$$\begin{aligned}\frac{di}{dt} &= \frac{Gm_t \sqrt{1-e^2} r_t}{an(1+e \cos f)} \left(\frac{S_{(t,A,i)}}{r_t^3} + \frac{S_{(t,B,i)}}{r_{t,B}^3} \right) \\ &\quad + \sum_{j=4}^N \left[\left(\frac{di}{dt} \right)_{j,A} + \left(\frac{di}{dt} \right)_{j,B} \right],\end{aligned}\quad (51)$$

$$\begin{aligned}\frac{d\Omega}{dt} &= \frac{Gm_t \sqrt{1-e^2} r_t}{an(1+e \cos f)} \left(\frac{S_{(t,A,\Omega)}}{r_t^3} + \frac{S_{(t,B,\Omega)}}{r_{t,B}^3} \right) \\ &\quad + \sum_{j=4}^N \left[\left(\frac{d\Omega}{dt} \right)_{j,A} + \left(\frac{d\Omega}{dt} \right)_{j,B} \right],\end{aligned}\quad (52)$$

$$\frac{d\omega}{dt} = \frac{Gm_t \sqrt{1-e^2} r_t}{2aen(1+e \cos f)} \left(\frac{S_{(t,A,\omega)}}{r_t^3} + \frac{S_{(t,B,\omega)}}{r_{t,B}^3} \right)$$

$$+ \sum_{j=4}^N \left[\left(\frac{d\omega}{dt} \right)_{j,A} + \left(\frac{d\omega}{dt} \right)_{j,B} \right], \quad (53)$$

$$\frac{df}{dt} = \frac{n(1 + e \cos f)^2}{(1 - e^2)^{3/2}} - \frac{d\omega}{dt} - \cos i \frac{d\Omega}{dt}. \quad (54)$$

where all contributions from the $j \geq 4$ bodies are given in equations (10)-(19). The auxiliary S variables, defined below, are particularly useful ways to characterize the relative contributions from both the A and B terms. I express the S variables alternatively in terms of f and f_j to facilitate later study of configurations when the secondary is both interior and exterior to the tertiary. I find

$$S_{(j,A,a)} = eD_{j,1} + D_{j,1} \cos f + D_{j,2} \sin f \quad (55)$$

$$= D_{j,5} \cos f_j - D_{j,6} \sin f_j, \quad (56)$$

$$S_{(j,A,e)} = 3D_{j,1} + 2eD_{j,2} \sin f + 4eD_{j,1} \cos f \\ + D_{j,2} \sin 2f + D_{j,1} \cos 2f \quad (57)$$

$$= D_{j,7} \cos f_j - D_{j,8} \sin f_j, \quad (58)$$

$$S_{(j,A,i)} = \sin i \sin \chi_j \{ \sin f \sin \omega - \cos f \cos \omega \} \quad (59)$$

$$= \sin i \cos (f + \omega) \{ \cos (\Omega - \varpi_j) \sin f_j - \sin (\Omega - \varpi_j) \cos f_j \}, \quad (60)$$

$$S_{(j,A,\Omega)} = -\sin \chi_j \{ \sin \omega \cos f + \cos \omega \sin f \} \quad (61)$$

$$= \sin (f + \omega) \{ \cos (\Omega - \varpi_j) \sin f_j - \sin (\Omega - \varpi_j) \cos f_j \}, \quad (62)$$

$$S_{(j,A,\omega)} = 3D_{j,2} + (2e \cos i \cos \omega \sin \chi_j) \sin f \\ + (2e \cos \omega \cos \chi_j) \cos f - D_{j,2} \cos 2f + D_{j,1} \sin 2f \quad (63)$$

$$= D_{j,10} \sin f_j - D_{j,9} \cos f_j \quad (64)$$

and

$$S_{(j,B,a)} = -S_{(j,A,a)} - \left(\frac{r}{r_j} \right) e \sin f, \quad (65)$$

$$S_{(j,B,e)} = -S_{(j,A,e)} - \left(\frac{r}{r_j} \right) (2 \sin f + e \sin 2f), \quad (66)$$

$$S_{(j,B,i)} = -S_{(j,A,i)}, \quad (67)$$

$$S_{(j,B,\Omega)} = -S_{(j,A,\Omega)}, \quad (68)$$

$$S_{(j,B,\omega)} = -S_{(j,A,\omega)} + \left(\frac{r}{r_j} \right) (2 \cos f + 2e \cos^2 f), \quad (69)$$

with

$$D_{j,5} = C_1 \cos i \sin (\Omega - \varpi_j) + C_2 \cos (\Omega - \varpi_j), \quad (70)$$

$$D_{j,6} = C_1 \cos i \cos (\Omega - \varpi_j) - C_2 \sin (\Omega - \varpi_j), \quad (71)$$

$$D_{j,7} = C_6 \cos i \sin (\Omega - \varpi_j) + C_5 \cos (\Omega - \varpi_j), \quad (72)$$

$$D_{j,8} = C_6 \cos i \cos (\Omega - \varpi_j) - C_5 \sin (\Omega - \varpi_j), \quad (73)$$

$$D_{j,9} = C_8 \cos i \sin (\Omega - \varpi_j) - C_7 \cos (\Omega - \varpi_j), \quad (74)$$

$$D_{j,10} = C_8 \cos i \cos (\Omega - \varpi_j) + C_7 \sin (\Omega - \varpi_j). \quad (75)$$

The lack of an extra term on the RHS of both equations (67) and (68) explains why the inclination and longitude of ascending node are more simply expressed analytically than the eccentricity and longitude of pericentre. The variation of the inclination and longitude of ascending node have a symmetry about both perturbative terms (equations 2-3) that is lacking from the eccentricity and argument of pericentre.

5.2 When all massive bodies are coplanar

If, however, all bodies in the system except perhaps the secondary are coplanar and the pericentres of the coplanar bodies (from elliptical, parabolic or hyperbolic orbits) are all measured with respect to the same reference direction, then the equations of motion reduce to

$$\frac{da}{dt} = \sum_{j=3}^N \frac{2Gm_j r_j}{n\sqrt{1-e^2}} \left(\frac{S_{(j,A,a)}}{r_j^3} + \frac{S_{(j,B,a)}}{r_{j,B}^3} \right), \quad (76)$$

$$\frac{de}{dt} = \sum_{j=3}^N \frac{Gm_j \sqrt{1-e^2} r_j}{2an(1+e \cos f)} \left(\frac{S_{(j,A,e)}}{r_j^3} + \frac{S_{(j,B,e)}}{r_{j,B}^3} \right), \quad (77)$$

$$\frac{di}{dt} = \sum_{j=3}^N \frac{Gm_j \sqrt{1-e^2} r_j}{an(1+e \cos f)} \left(\frac{S_{(j,A,i)}}{r_j^3} + \frac{S_{(j,B,i)}}{r_{j,B}^3} \right), \quad (78)$$

$$\frac{d\Omega}{dt} = \sum_{j=3}^N \frac{Gm_j \sqrt{1-e^2} r_j}{an(1+e \cos f)} \left(\frac{S_{(j,A,\Omega)}}{r_j^3} + \frac{S_{(j,B,\Omega)}}{r_{j,B}^3} \right), \quad (79)$$

$$\frac{d\omega}{dt} = \sum_{j=3}^N \frac{Gm_j \sqrt{1-e^2} r_j}{2aen(1+e \cos f)} \left(\frac{S_{(j,A,\omega)}}{r_j^3} + \frac{S_{(j,B,\omega)}}{r_{j,B}^3} \right), \quad (80)$$

$$\frac{df}{dt} = \frac{n(1+e \cos f)^2}{(1-e^2)^{3/2}} - \frac{d\omega}{dt} - \cos i \frac{d\Omega}{dt}. \quad (81)$$

The remainder of the paper utilizes equations (76)-(81) or reductions of these equations.

5.2.1 Preparations for averaging

The equations may be recast in a form that will be useful when one (later) wishes to obtain averaged effects. Typically, averaging can be performed only

if the quantity $r_{3,B}$ is expanded about a dimensionless ratio $\alpha_{sc} \equiv r/r_t$ or $\alpha_{sf} \equiv r_t/r$ in a power series, where the subscripts indicate if the secondary is *close* or *far* from the primary. Hence, the following identity is useful.

$$(1 - \kappa_t \alpha_{sc} + \alpha_{sc}^2)^{-3/2} = \sum_{u=0}^{\infty} \binom{\frac{1}{2} + u}{\frac{1}{2}} \alpha_{sc}^u [\kappa_t - \alpha_{sc}]^u \quad (82)$$

Consequently, the equations of motion may be expressed as a power series in α_{sc} or α_{sf} . In the former case, by using equations (65)-(69), I obtain

$$\begin{aligned} \frac{da}{dt} = & \sum_{j=3}^N \frac{-2Gm_j}{n\sqrt{1-e^2}r_j^2} \left[S_{(j,A,a)} \sum_{u=1}^{\infty} \binom{\frac{1}{2} + u}{\frac{1}{2}} \alpha_{sc}^u (\kappa_j - \alpha_{sc})^u \right. \\ & \left. + e \sin f \sum_{u=0}^{\infty} \binom{\frac{1}{2} + u}{\frac{1}{2}} \alpha_{sc}^{u+1} (\kappa_j - \alpha_{sc})^u \right], \end{aligned} \quad (83)$$

$$\begin{aligned} \frac{de}{dt} = & \sum_{j=3}^N \frac{-Gm_j\sqrt{1-e^2}}{2an(1+e\cos f)r_j^2} \left[S_{(j,A,e)} \sum_{u=1}^{\infty} \binom{\frac{1}{2} + u}{\frac{1}{2}} \alpha_{sc}^u (\kappa_j - \alpha_{sc})^u \right. \\ & \left. + (2\sin f + e\sin 2f) \sum_{u=0}^{\infty} \binom{\frac{1}{2} + u}{\frac{1}{2}} \alpha_{sc}^{u+1} (\kappa_j - \alpha_{sc})^u \right], \end{aligned} \quad (84)$$

$$\frac{di}{dt} = \sum_{j=3}^N \frac{-Gm_j\sqrt{1-e^2}}{an(1+e\cos f)r_j^2} S_{(j,A,i)} \sum_{u=1}^{\infty} \binom{\frac{1}{2} + u}{\frac{1}{2}} \alpha_{sc}^u (\kappa_j - \alpha_{sc})^u, \quad (85)$$

$$\frac{d\Omega}{dt} = \sum_{j=3}^N \frac{-Gm_j\sqrt{1-e^2}}{an(1+e\cos f)r_j^2} S_{(j,A,\Omega)} \sum_{u=1}^{\infty} \binom{\frac{1}{2} + u}{\frac{1}{2}} \alpha_{sc}^u (\kappa_j - \alpha_{sc})^u, \quad (86)$$

$$\begin{aligned} \frac{d\omega}{dt} = & \sum_{j=3}^N \frac{Gm_j\sqrt{1-e^2}}{2aen(1+e\cos f)r_j^2} \left[-S_{(j,A,\omega)} \sum_{u=1}^{\infty} \binom{\frac{1}{2} + u}{\frac{1}{2}} \alpha_{sc}^u (\kappa_j - \alpha_{sc})^u \right. \\ & \left. + 2(\cos f + e\cos^2 f) \sum_{u=0}^{\infty} \binom{\frac{1}{2} + u}{\frac{1}{2}} \alpha_{sc}^{u+1} (\kappa_j - \alpha_{sc})^u \right]. \end{aligned} \quad (87)$$

Note the difference in the starting summation indices. The power series in α_{sf} is not particularly useful here because typically when the secondary is exterior to the tertiary, one no longer wishes to measure the secondary's orbit elements with respect to the primary.

6 Full coplanarity of all bodies

If I now impose coplanarity on the secondary as well, then the equations are greatly simplified. I set $i = 0^\circ$ and $\varpi = \omega + \Omega$. Consequently, the auxiliary *planar* C variables, denoted by C^P , take on the same form except with the substitution $\omega \rightarrow \varpi$. The two exceptions are $C_3^P = -\cos(f + \varpi)$ and $C_4^P = \sin(f + \varpi)$. Hence,

$$x_{j,B} = r_j \cos(f_j + \varpi_j) - r \cos(f + \varpi), \quad (88)$$

$$y_{j,B} = r_j \sin(f_j + \varpi_j) - r \sin(f + \varpi), \quad (89)$$

$$z_{j,B} = 0, \quad (90)$$

with

$$x_j = r_j \cos(f_j + \varpi_j), \quad (91)$$

$$y_j = r_j \sin(f_j + \varpi_j), \quad (92)$$

$$z_j = 0, \quad (93)$$

and

$$\chi_j^P = \varpi - f_j - \varpi_j, \quad (94)$$

$$\kappa_j^P = 2 \cos(f + \chi_j^P) \quad (95)$$

$$= 2 (D_{j,2}^P \cos f - D_{j,1}^P \sin f) \quad (96)$$

$$= 2 (D_{j,3}^P \cos f_j + D_{j,4}^P \sin f_j) \quad (97)$$

such that

$$D_{j,1}^P = \sin \chi_j^P, \quad (98)$$

$$D_{j,2}^P = \cos \chi_j^P, \quad (99)$$

$$D_{j,3}^P = \cos(f + \varpi - \varpi_j), \quad (100)$$

$$D_{j,4}^P = \sin(f + \varpi - \varpi_j). \quad (101)$$

The fully coplanar equations of motion, which are denoted with a superscript P, become

$$\left(\frac{da}{dt}\right)_{j,A}^P = \frac{2Gm_j}{n\sqrt{1-e^2}r_j^3} [x_j C_2^P - y_j C_1^P], \quad (102)$$

$$\left(\frac{de}{dt}\right)_{j,A}^P = \frac{Gm_j\sqrt{1-e^2}}{2an(1+e\cos f)r_j^3} [x_j C_5^P - y_j C_6^P], \quad (103)$$

$$\left(\frac{di}{dt}\right)_{j,A}^P = 0, \quad (104)$$

$$\left(\frac{d\varpi}{dt}\right)_{j,A}^P = \frac{Gm_j\sqrt{1-e^2}}{2aen(1+e\cos f)r_j^3} [x_j C_7^P + y_j C_9^P], \quad (105)$$

and

$$\left(\frac{da}{dt}\right)_{j,B}^P = \frac{2Gm_j}{n\sqrt{1-e^2}r_{j,B}^3} \times \left[\frac{-ae(1-e^2)\sin f}{1+e\cos f} - x_j C_2^P + y_j C_1^P \right], \quad (106)$$

$$\left(\frac{de}{dt}\right)_{j,B}^P = \frac{Gm_j\sqrt{1-e^2}}{2an(1+e\cos f)r_{j,B}^3} \times [-2a(1-e^2)\sin f - x_j C_5^P + y_j C_6^P], \quad (107)$$

$$\left(\frac{di}{dt}\right)_{j,B}^P = 0, \quad (108)$$

$$\left(\frac{d\varpi}{dt}\right)_{j,B}^P = \frac{Gm_j\sqrt{1-e^2}}{2aen(1+e\cos f)r_{j,B}^3} \times [2a(1-e^2)\cos f - x_j C_7^P - y_j C_9^P]. \quad (109)$$

with

$$\left(\frac{df}{dt}\right)^P = \frac{n(1+e\cos f)^2}{(1-e^2)^{3/2}} - \left(\frac{d\varpi}{dt}\right)_{j,A}^P - \left(\frac{d\varpi}{dt}\right)_{j,B}^P. \quad (110)$$

I find here, as in Veras & Evans (2013), that $(d\Omega/dt)^P \neq 0$ even though the orbits remain coplanar. This feature has no physical consequence, but is likely mathematically important to include if ω was kept in the equations instead of ϖ .

6.1 Preparations for averaging

I eliminate the C^P terms and reexpress the terms in square brackets for the fully coplanar equations of motion as

$$\left(\frac{da}{dt}\right)^P = \sum_{j=3}^N \frac{-2Gm_j}{n\sqrt{1-e^2}r_j^2} \left[S_{(j,A,a)}^P \sum_{u=1}^{\infty} \left(\frac{\frac{1}{2}+u}{\frac{1}{2}}\right) \alpha_{sc}^u (\kappa_j^P - \alpha_{sc})^u + e \sin f \sum_{u=0}^{\infty} \left(\frac{\frac{1}{2}+u}{\frac{1}{2}}\right) \alpha_{sc}^{u+1} (\kappa_j^P - \alpha_{sc})^u \right], \quad (111)$$

$$\left(\frac{de}{dt}\right)^P = \sum_{j=3}^N \frac{-Gm_j\sqrt{1-e^2}}{2an(1+e\cos f)r_j^2} \left[S_{(j,A,e)}^P \sum_{u=1}^{\infty} \left(\frac{\frac{1}{2}+u}{\frac{1}{2}}\right) \alpha_{sc}^u (\kappa_j^P - \alpha_{sc})^u + (2\sin f + e \sin 2f) \sum_{u=0}^{\infty} \left(\frac{\frac{1}{2}+u}{\frac{1}{2}}\right) \alpha_{sc}^{u+1} (\kappa_j^P - \alpha_{sc})^u \right], \quad (112)$$

$$\begin{aligned} \left(\frac{d\varpi}{dt}\right)^P &= \sum_{j=3}^N \frac{Gm_j \sqrt{1-e^2}}{2aen(1+e\cos f)r_j^2} \left[-S_{(j,A,\varpi)}^P \sum_{u=1}^{\infty} \left(\frac{\frac{1}{2}+u}{\frac{1}{2}}\right) \alpha_{sc}^u (\kappa_j^P - \alpha_{sc})^u \right. \\ &\quad \left. + 2(\cos f + e\cos^2 f) \sum_{u=0}^{\infty} \left(\frac{\frac{1}{2}+u}{\frac{1}{2}}\right) \alpha_{sc}^{u+1} (\kappa_j^P - \alpha_{sc})^u \right], \end{aligned} \quad (113)$$

where

$$S_{(j,A,a)}^P = eD_{j,1}^P + D_{j,1}^P \cos f + D_{j,2}^P \sin f \quad (114)$$

$$= D_{j,5}^P \cos f_j - D_{j,6}^P \sin f_j, \quad (115)$$

$$\begin{aligned} S_{(j,A,e)}^P &= 3D_{j,1}^P + 4eD_{j,1}^P \cos f + D_{j,1}^P \cos 2f \\ &\quad + 2eD_{j,2}^P \sin f + D_{j,2}^P \sin 2f \end{aligned} \quad (116)$$

$$= D_{j,7}^P \cos f_j - D_{j,8}^P \sin f_j, \quad (117)$$

$$S_{(j,A,\varpi)}^P = 3D_{j,2}^P + 2eD_{j,2}^P \cos f - D_{j,2}^P \cos 2f + D_{j,1}^P \sin 2f \quad (118)$$

$$= D_{j,12}^P \cos f_j - D_{j,11}^P \sin f_j, \quad (119)$$

and

$$S_{(j,B,a)}^P = -S_{(j,A,a)}^P - \left(\frac{r}{r_j}\right) e \sin f, \quad (120)$$

$$S_{(j,B,e)}^P = -S_{(j,A,e)}^P - \left(\frac{r}{r_t}\right) (2 \sin f + e \sin 2f), \quad (121)$$

$$S_{(j,B,\varpi)}^P = -S_{(j,A,\varpi)}^P + \left(\frac{r}{r_t}\right) (2 \cos f + 2e \cos^2 f), \quad (122)$$

with

$$D_{j,5}^P = e \sin(\varpi - \varpi_j) + \sin(f + \varpi - \varpi_j), \quad (123)$$

$$D_{j,6}^P = e \cos(\varpi - \varpi_j) + \cos(f + \varpi - \varpi_j), \quad (124)$$

$$\begin{aligned} D_{j,7}^P &= 3 \sin(\varpi - \varpi_j) + 3e \sin(f + \varpi - \varpi_j) \\ &\quad + \sin(2f + \varpi - \varpi_j) - e \sin(f - \varpi + \varpi_j), \end{aligned} \quad (125)$$

$$\begin{aligned} D_{j,8}^P &= 3 \cos(\varpi - \varpi_j) + 3e \cos(f + \varpi - \varpi_j) \\ &\quad + \cos(2f + \varpi - \varpi_j) + e \cos(f - \varpi + \varpi_j), \end{aligned} \quad (126)$$

$$D_{j,11} = \sin(2f + \varpi - \varpi_j) - (3 + 2e \cos f) \sin(\varpi - \varpi_j), \quad (127)$$

$$D_{j,12} = -\cos(2f + \varpi - \varpi_j) + (3 + 2e \cos f) \cos(\varpi - \varpi_j). \quad (128)$$

7 Three bodies only

Here I briefly place three-body systems in context of the equations of motion already presented, before describing these systems in greater detail in later sections. First, I note that the Tisserand parameter does not apply in the averaged systems which I describe later.

7.1 An inclined and circular tertiary

The equations of motion for this system are equations (76)-(81), or, alternatively, (83)-(87) plus equation (81). If I assume that the tertiary is on a circular (bound) orbit, regardless of the orientation of the secondary orbit, then I can both simplify the equations ($e_t = 0$) and introduce an additional constraint on the system through the Tisserand parameter, T . Then the Tisserand parameter is conserved such that

$$\begin{aligned} T &= \frac{a_t}{a} + 2\sqrt{\frac{a(1-e^2)}{a_t}} \cos i \\ &= \frac{1}{\alpha_{sc}} \left(\frac{1-e^2}{1+e \cos f} \right) + 2 \cos i \sqrt{\alpha_{sc}(1+e \cos f)} \end{aligned} \quad (129)$$

which can help eliminate a , e , i or f from the equations of motion. Further, because T may be expressed in terms of α_{sc} , averaging may be facilitated.

7.2 A coplanar and circular tertiary

Here the equations of motion are equivalent to the fully coplanar equations (equations 102-110, or 111-113 plus 110) except that I may also use the Tisserand parameter (equation 129) with $i = 0$ to eliminate one variable.

In this system, $r_t = a_t$, and equations (40)-(41) hold, except with

$$\kappa_t = 2 \cos(f + f_j + \varpi) \quad (130)$$

$$= 2 [\cos(f_j + \varpi) \cos f - \sin(f_j + \varpi) \sin f], \quad (131)$$

$$= 2 [\cos(f + \varpi) \cos f_j - \sin(f + \varpi) \sin f_j]. \quad (132)$$

8 Shifting the reference origin

Until now, I have assumed that the secondary's orbital elements are measured with respect to the centre of the primary. This assumption generally yields useful elements when all other massive bodies in the system are further away from the primary than the secondary ⁵.

⁵ An exception might be a Trojan asteroid of a planet.

However, if the secondary orbits a cluster of massive bodies, then measuring the orbital elements with respect to just one of those bodies, like the primary, will yield unhelpful relations. The high reflex motion of the primary will make the secondary appear to oscillate between elliptic and hyperbolic orbits, in many cases improperly giving the impression that the system is unstable. The more natural way to trace the secondary's motion is to compute the orbital elements with respect to the centre of mass of the cluster.

Deriving these elements requires me to express the equation of motion of the secondary and the centre of mass of the other bodies in the same form as in equation (1). Subsequently, I could perform the same perturbation analysis as in Sections 4-7. However, there is a shortcut that enables me to derive the new equations more elegantly.

8.1 General 3-body equations of motion

To proceed, I restrict the derivation to three bodies here, and describe the general N -body case in Appendix A. Figure 2 shows the primary ("p"), secondary ("s") and tertiary ("t"), where $m_s = 0$, $m_p \neq 0$ and $m_t \neq 0$. I need to convert equation (1) into a similar equation for \mathbf{s} . I have

$$\mathbf{s} = -\mathbf{s}_p - \mathbf{r} = -\mathbf{s}_t + \mathbf{r}_t - \mathbf{r} = -\left(\frac{m_p}{m_p + m_t}\right) \mathbf{r}_t + \mathbf{r}_t - \mathbf{r} \quad (133)$$

so that insertion into equation (1) yields

$$\begin{aligned} & -\frac{d^2 \mathbf{s}}{dt^2} + \left(\frac{m_t}{m_p + m_t}\right) \frac{d^2 \mathbf{r}_t}{dt^2} + Gm_p \left[\frac{-\mathbf{s} + \left(\frac{m_t}{m_p + m_t}\right) \mathbf{r}_t}{\left|-\mathbf{s} + \left(\frac{m_t}{m_p + m_t}\right) \mathbf{r}_t\right|^3} \right] \\ & = Gm_t \left[\frac{\mathbf{r}_t + \mathbf{s} - \left(\frac{m_t}{m_p + m_t}\right) \mathbf{r}_t}{\left|\mathbf{r}_t + \mathbf{s} - \left(\frac{m_t}{m_p + m_t}\right) \mathbf{r}_t\right|^3} \right] - Gm_t \frac{\mathbf{r}_t}{r_t^3}. \end{aligned} \quad (134)$$

In order to eliminate $d^2 \mathbf{r}_t / dt^2$ from equation (134), I use the properties of the two body problem (as the secondary has no mass) to write

$$\frac{d^2 \mathbf{r}_t}{dt^2} + G(m_p + m_t) \frac{\mathbf{r}_t}{r_t^3} = 0 \quad (135)$$

so that by adding and subtracting the same term below, I finally obtain the desired form

$$\frac{d^2 \mathbf{s}}{dt^2} = -\frac{G(m_p + m_t) \mathbf{s}}{s^3} + Gm_p \left[\frac{\left(\frac{m_t}{m_p + m_t}\right) \mathbf{r}_t - \mathbf{s}}{\left|\left(\frac{m_t}{m_p + m_t}\right) \mathbf{r}_t - \mathbf{s}\right|^3} \right] \quad (136)$$

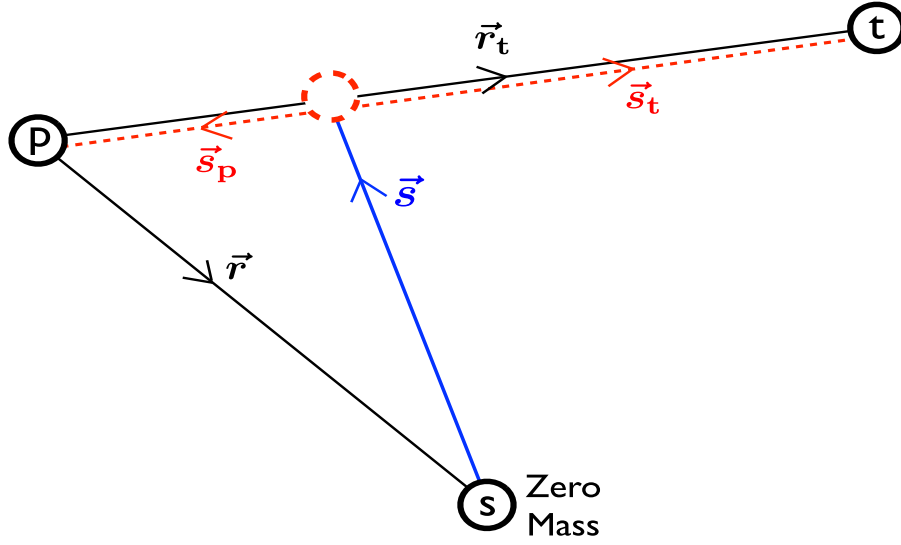


Fig. 2 Vectors used in the derivation of equation (138) to obtain the equations of motion with orbital elements that are measured with respect to the centre of mass of the primary and tertiary.

$$- Gm_t \left[\frac{\left(\frac{m_p}{m_p + m_t} \right) \mathbf{r}_t + \mathbf{s}}{\left| \left(\frac{m_p}{m_p + m_t} \right) \mathbf{r}_t + \mathbf{s} \right|^3} \right] + \frac{G(m_p + m_t) \mathbf{s}}{s^3}, \quad (137)$$

which can be compared to equation (1).

This form demonstrates that the perturbation Δ' to the two-body problem in the barycentric reference frame is composed of three terms, all variations of $\Delta_{j,B}$ (equation 3). I denote the orbital elements in this reference frame with primes, and let β and β' represent placeholders for any of $(a, e, i, \Omega, \omega, f)$ and $(a', e', i', \Omega', \omega', f')$ respectively. Then the barycentric equations of motion in orbital elements are given fully by

$$\begin{aligned} \frac{d\beta'}{dt} &= \frac{m_p}{m_t} \left(\frac{d\beta}{dt} \right)_{t,B} \bigg|_{\mathbf{r}_t \rightarrow \frac{m_t}{m_p + m_t} \mathbf{r}_t}^{(a,e,i,\Omega,\omega,f) \rightarrow (a',e',i',\Omega',\omega',f')} \\ &+ \left(\frac{d\beta}{dt} \right)_{t,B} \bigg|_{\mathbf{r}_t \rightarrow \frac{-m_p}{m_p + m_t} \mathbf{r}_t}^{(a,e,i,\Omega,\omega,f) \rightarrow (a',e',i',\Omega',\omega',f')} \\ &- \left(\frac{m_p + m_t}{m_t} \right) \left(\frac{d\beta}{dt} \right)_{t,B} \bigg|_{\mathbf{r}_t \rightarrow 0}^{(a,e,i,\Omega,\omega,f) \rightarrow (a',e',i',\Omega',\omega',f')}. \end{aligned} \quad (138)$$

I now write out this expression for each transformed orbital element through equations (41), (49)-(53) and (65)-(69) first by rewriting the unprimed B term expressions as

$$\begin{aligned} \left(\frac{da}{dt}\right)_{t,B} &= \frac{2Gm_t S_{(t,B,a)}}{n\sqrt{1-e^2}} \left(\frac{r_t}{r_{t,B}^3}\right) \\ &= \frac{2Gm_t (-S_{(t,A,a)}\alpha_{sf} - e \sin f)}{n\sqrt{1-e^2}r^2} (1 - \kappa_t\alpha_{sf} + \alpha_{sf}^2)^{-\frac{3}{2}}, \end{aligned} \quad (139)$$

$$\begin{aligned} \left(\frac{de}{dt}\right)_{t,B} &= \frac{Gm_t\sqrt{1-e^2}S_{(t,B,e)}}{2an(1+e\cos f)} \left(\frac{r_t}{r_{t,B}^3}\right) \\ &= \frac{Gm_t\sqrt{1-e^2}(-S_{(t,A,e)}\alpha_{sf} - 2\sin f - e\sin 2f)}{2an(1+e\cos f)r^2} \\ &\quad \times (1 - \kappa_t\alpha_{sf} + \alpha_{sf}^2)^{-\frac{3}{2}}, \end{aligned} \quad (140)$$

$$\begin{aligned} \left(\frac{di}{dt}\right)_{t,B} &= \frac{Gm_t\sqrt{1-e^2}S_{(t,B,i)}}{an(1+e\cos f)} \left(\frac{r_t}{r_{t,B}^3}\right) \\ &= -\frac{Gm_t\sqrt{1-e^2}S_{(t,A,i)}\alpha_{sf}}{an(1+e\cos f)r^2} (1 - \kappa_t\alpha_{sf} + \alpha_{sf}^2)^{-\frac{3}{2}}, \end{aligned} \quad (141)$$

$$\begin{aligned} \left(\frac{d\Omega}{dt}\right)_{t,B} &= \frac{Gm_t\sqrt{1-e^2}S_{(t,B,\Omega)}}{an(1+e\cos f)} \left(\frac{r_t}{r_{t,B}^3}\right) \\ &= -\frac{Gm_t\sqrt{1-e^2}S_{(t,A,\Omega)}\alpha_{sf}}{an(1+e\cos f)r^2} (1 - \kappa_t\alpha_{sf} + \alpha_{sf}^2)^{-\frac{3}{2}}, \end{aligned} \quad (142)$$

$$\begin{aligned} \left(\frac{d\omega}{dt}\right)_{t,B} &= \frac{Gm_t\sqrt{1-e^2}S_{(t,B,\omega)}}{2aen(1+e\cos f)} \left(\frac{r_t}{r_{t,B}^3}\right) \\ &= \frac{Gm_t\sqrt{1-e^2}(-S_{(t,A,\omega)}\alpha_{sf} + 2\cos f + 2e\cos^2 f)}{2aen(1+e\cos f)r^2} \\ &\quad \times (1 - \kappa_t\alpha_{sf} + \alpha_{sf}^2)^{-\frac{3}{2}}. \end{aligned} \quad (143)$$

The forms in equations (139)-(143) facilitate the derivation of the equations of motion, which are:

$$\begin{aligned} \frac{da'}{dt} &= \frac{2Gm_p \left(-S_{(t,A,a')}\alpha_{sf} \left(\frac{m_t}{m_p+m_t}\right) - e' \sin f'\right)}{n'\sqrt{1-e'^2}r'^2} \\ &\quad \times \left(1 - \kappa'_t\alpha_{sf} \left(\frac{m_t}{m_p+m_t}\right) + \alpha_{sf}^2 \left(\frac{m_t}{m_p+m_t}\right)^2\right)^{-3/2} \\ &\quad + \frac{2Gm_t \left(S_{(t,A,a')}\alpha_{sf} \left(\frac{m_p}{m_p+m_t}\right) - e' \sin f'\right)}{n'\sqrt{1-e'^2}r'^2} \\ &\quad \times \left(1 + \kappa'_t\alpha_{sf} \left(\frac{m_p}{m_p+m_t}\right) + \alpha_{sf}^2 \left(\frac{m_p}{m_p+m_t}\right)^2\right)^{-3/2} \end{aligned}$$

$$- \frac{2G(m_p + m_t)(-e' \sin f')}{n' \sqrt{1 - e'^2} r'^2} \quad (144)$$

$$\begin{aligned} \frac{de'}{dt} = & \frac{Gm_p \sqrt{1 - e'^2} \left(-S_{(t,A,e')} \alpha_{sf} \left(\frac{m_t}{m_p + m_t} \right) - 2 \sin f' - e' \sin 2f' \right)}{2a'n' (1 + e' \cos f') r'^2} \\ & \times \left(1 - \kappa'_t \alpha_{sf} \left(\frac{m_t}{m_p + m_t} \right) + \alpha_{sf}^2 \left(\frac{m_t}{m_p + m_t} \right)^2 \right)^{-3/2} \\ & + \frac{Gm_t \sqrt{1 - e'^2} \left(S_{(t,A,e')} \alpha_{sf} \left(\frac{m_p}{m_p + m_t} \right) - 2 \sin f' - e' \sin 2f' \right)}{2a'n' (1 + e' \cos f') r'^2} \\ & \times \left(1 + \kappa'_t \alpha_{sf} \left(\frac{m_p}{m_p + m_t} \right) + \alpha_{sf}^2 \left(\frac{m_p}{m_p + m_t} \right)^2 \right)^{-3/2} \\ & - \frac{G(m_p + m_t) \sqrt{1 - e'^2} (-2 \sin f' - e' \sin 2f')}{2a'n' (1 + e' \cos f') r'^2}, \end{aligned} \quad (145)$$

$$\begin{aligned} \frac{di'}{dt} = & - \frac{Gm_t m_p \sqrt{1 - e'^2} S_{(t,A,i')} \alpha_{sf}}{a'n' (1 + e' \cos f') r'^2 (m_p + m_t)} \\ & \times \left(1 - \kappa'_t \alpha_{sf} \left(\frac{m_t}{m_p + m_t} \right) + \alpha_{sf}^2 \left(\frac{m_t}{m_p + m_t} \right)^2 \right)^{-3/2} \\ & + \frac{Gm_t m_p \sqrt{1 - e'^2} S_{(t,A,i')} \alpha_{sf}}{a'n' (1 + e' \cos f') r'^2 (m_p + m_t)} \\ & \times \left(1 + \kappa'_t \alpha_{sf} \left(\frac{m_p}{m_p + m_t} \right) + \alpha_{sf}^2 \left(\frac{m_p}{m_p + m_t} \right)^2 \right)^{-3/2}, \end{aligned} \quad (146)$$

$$\begin{aligned} \frac{d\Omega'}{dt} = & - \frac{Gm_t m_p \sqrt{1 - e'^2} S_{(t,A,\Omega')} \alpha_{sf}}{a'n' (1 + e' \cos f') r'^2 (m_p + m_t)} \\ & \times \left(1 - \kappa'_t \alpha_{sf} \left(\frac{m_t}{m_p + m_t} \right) + \alpha_{sf}^2 \left(\frac{m_t}{m_p + m_t} \right)^2 \right)^{-3/2} \\ & + \frac{Gm_t m_p \sqrt{1 - e'^2} S_{(t,A,\Omega')} \alpha_{sf}}{a'n' (1 + e' \cos f') r'^2 (m_p + m_t)} \\ & \times \left(1 + \kappa'_t \alpha_{sf} \left(\frac{m_p}{m_p + m_t} \right) + \alpha_{sf}^2 \left(\frac{m_p}{m_p + m_t} \right)^2 \right)^{-3/2}, \end{aligned} \quad (147)$$

$$\begin{aligned} \frac{d\omega'}{dt} = & \frac{Gm_p \sqrt{1 - e'^2} \left(-S_{(t,A,\omega')} \alpha_{sf} \left(\frac{m_t}{m_p + m_t} \right) + 2 \cos f' + 2e' \cos^2 f' \right)}{2a'e'n' (1 + e' \cos f') r'^2} \\ & \times \left(1 - \kappa'_t \alpha_{sf} \left(\frac{m_t}{m_p + m_t} \right) + \alpha_{sf}^2 \left(\frac{m_t}{m_p + m_t} \right)^2 \right)^{-3/2} \end{aligned}$$

$$\begin{aligned}
& + \frac{Gm_t \sqrt{1-e'^2} \left(S_{(t,A,\omega')} \alpha_{sf} \left(\frac{m_p}{m_p+m_t} \right) + 2 \cos f' + 2e' \cos^2 f' \right)}{2a'e'n'(1+e' \cos f') r'^2} \\
& \times \left(1 + \kappa'_t \alpha_{sf} \left(\frac{m_p}{m_p+m_t} \right) + \alpha_{sf}^2 \left(\frac{m_p}{m_p+m_t} \right)^2 \right)^{-3/2} \\
& - \frac{G(m_p+m_t) \sqrt{1-e'^2} (2 \cos f' + 2e' \cos^2 f')}{2a'e'n'(1+e' \cos f') r'^2}, \tag{148}
\end{aligned}$$

$$\frac{df'}{dt} = \frac{n'(1+e' \cos f')^2}{(1-e'^2)^{3/2}} - \frac{d\omega'}{dt} - \cos i' \frac{d\Omega'}{dt}. \tag{149}$$

Equations (144)-(149) are the complete equations of motion in orbital elements for a secondary's orbit with respect to the barycentre of the primary and tertiary. Note that the orbital elements in the S variables are primed.

8.2 Preparations for averaging

By inspection, one may note that all terms to zeroth order in α_{sf} for all of the equations cancel. Therefore, the time evolution of the elements are always dependent on the ratio of semimajor axes. When expressed as a power series in semimajor axis ratio, the equations of motion become

$$\begin{aligned}
\frac{da'}{dt} = & - \frac{2GS_{(t,A,a')} \alpha_{sf}}{n' \sqrt{1-e'^2} r'^2} \left(\frac{m_p m_t}{m_p+m_t} \right) \\
& \times \sum_{u=0}^{\infty} \left(\frac{\frac{1}{2}+u}{\frac{1}{2}} \right) \left(\frac{\alpha_{sf}}{m_p+m_t} \right)^u \left\{ m_t^u \left[\kappa'_t - \left(\frac{m_t}{m_p+m_t} \right) \alpha_{sf} \right]^u \right. \\
& \left. - (-m_p)^u \left[\kappa'_t + \left(\frac{m_p}{m_p+m_t} \right) \alpha_{sf} \right]^u \right\} \\
& - \frac{2Ge' \sin f'}{n' \sqrt{1-e'^2} r'^2} \\
& \times \sum_{u=1}^{\infty} \left(\frac{\frac{1}{2}+u}{\frac{1}{2}} \right) \left(\frac{\alpha_{sf}}{m_p+m_t} \right)^u \left\{ m_p m_t^u \left[\kappa'_t - \left(\frac{m_t}{m_p+m_t} \right) \alpha_{sf} \right]^u \right. \\
& \left. + m_t (-m_p)^u \left[\kappa'_t + \left(\frac{m_p}{m_p+m_t} \right) \alpha_{sf} \right]^u \right\}, \tag{150}
\end{aligned}$$

$$\begin{aligned}
\frac{de'}{dt} = & - \frac{GS_{(t,A,e')} \sqrt{1-e'^2} \alpha_{sf}}{2a'n'(1+e' \cos f') r'^2} \left(\frac{m_p m_t}{m_p+m_t} \right) \\
& \times \sum_{u=0}^{\infty} \left(\frac{\frac{1}{2}+u}{\frac{1}{2}} \right) \left(\frac{\alpha_{sf}}{m_p+m_t} \right)^u \left\{ m_t^u \left[\kappa'_t - \left(\frac{m_t}{m_p+m_t} \right) \alpha_{sf} \right]^u \right.
\end{aligned}$$

$$\begin{aligned}
& - (-m_p)^u \left[\kappa'_t + \left(\frac{m_p}{m_p + m_t} \right) \alpha_{sf} \right]^u \Big\} \\
& - \frac{G\sqrt{1-e'^2} (2 \sin f' + e' \sin 2f')}{2a'n' (1 + e' \cos f') r'^2} \\
& \times \sum_{u=1}^{\infty} \left(\frac{\frac{1}{2} + u}{\frac{1}{2}} \right) \left(\frac{\alpha_{sf}}{m_p + m_t} \right)^u \left\{ m_p m_t^u \left[\kappa'_t - \left(\frac{m_t}{m_p + m_t} \right) \alpha_{sf} \right]^u \right. \\
& \left. + m_t (-m_p)^u \left[\kappa'_t + \left(\frac{m_p}{m_p + m_t} \right) \alpha_{sf} \right]^u \right\}, \tag{151}
\end{aligned}$$

$$\begin{aligned}
\frac{di'}{dt} &= - \frac{GS_{(t,A,i')} \sqrt{1-e'^2} \alpha_{sf}}{a'n' (1 + e' \cos f') r'^2} \left(\frac{m_p m_t}{m_p + m_t} \right) \\
& \times \sum_{u=0}^{\infty} \left(\frac{\frac{1}{2} + u}{\frac{1}{2}} \right) \left(\frac{\alpha_{sf}}{m_p + m_t} \right)^u \left\{ m_t^u \left[\kappa'_t - \left(\frac{m_t}{m_p + m_t} \right) \alpha_{sf} \right]^u \right. \\
& \left. - (-m_p)^u \left[\kappa'_t + \left(\frac{m_p}{m_p + m_t} \right) \alpha_{sf} \right]^u \right\}, \tag{152}
\end{aligned}$$

$$\begin{aligned}
\frac{d\Omega'}{dt} &= - \frac{GS_{(t,A,\Omega')} \sqrt{1-e'^2} \alpha_{sf}}{a'n' (1 + e' \cos f') r'^2} \left(\frac{m_p m_t}{m_p + m_t} \right) \\
& \times \sum_{u=0}^{\infty} \left(\frac{\frac{1}{2} + u}{\frac{1}{2}} \right) \left(\frac{\alpha_{sf}}{m_p + m_t} \right)^u \left\{ m_t^u \left[\kappa'_t - \left(\frac{m_t}{m_p + m_t} \right) \alpha_{sf} \right]^u \right. \\
& \left. - (-m_p)^u \left[\kappa'_t + \left(\frac{m_p}{m_p + m_t} \right) \alpha_{sf} \right]^u \right\}, \tag{153}
\end{aligned}$$

$$\begin{aligned}
\frac{d\omega'}{dt} &= - \frac{GS_{(t,A,\omega')} \sqrt{1-e'^2} \alpha_{sf}}{2a'e'n' (1 + e' \cos f') r'^2} \left(\frac{m_p m_t}{m_p + m_t} \right) \\
& \times \sum_{u=0}^{\infty} \left(\frac{\frac{1}{2} + u}{\frac{1}{2}} \right) \left(\frac{\alpha_{sf}}{m_p + m_t} \right)^u \left\{ m_t^u \left[\kappa'_t - \left(\frac{m_t}{m_p + m_t} \right) \alpha_{sf} \right]^u \right. \\
& \left. - (-m_p)^u \left[\kappa'_t + \left(\frac{m_p}{m_p + m_t} \right) \alpha_{sf} \right]^u \right\} \\
& + \frac{G\sqrt{1-e'^2} (2 \cos f' + 2e' \cos^2 f')}{2a'e'n' (1 + e' \cos f') r'^2} \\
& \times \sum_{u=1}^{\infty} \left(\frac{\frac{1}{2} + u}{\frac{1}{2}} \right) \left(\frac{\alpha_{sf}}{m_p + m_t} \right)^u \left\{ m_p m_t^u \left[\kappa'_t - \left(\frac{m_t}{m_p + m_t} \right) \alpha_{sf} \right]^u \right. \\
& \left. + m_t (-m_p)^u \left[\kappa'_t + \left(\frac{m_p}{m_p + m_t} \right) \alpha_{sf} \right]^u \right\}. \tag{154}
\end{aligned}$$

Note that the starting index on the first summation may be increased to 1 because the $u = 0$ term vanishes. Also,

$$\frac{df'}{dt} = \frac{n' (1 + e' \cos f')^2}{(1 - e'^2)^{3/2}} - \frac{d\omega'}{dt} - \cos i' \frac{d\Omega'}{dt}. \quad (155)$$

Regarding the fully coplanar equations of motion, by analogy with the transition from the general to the planar case in the reference frame of the primary, I can write

$$\left(\frac{da'}{dt} \right)^P = \frac{da'}{dt} \Big|_{S_{(t,A,a')} \rightarrow S_{(t,A,a')}^P}^{\kappa'_t \rightarrow \kappa'_t{}^P}, \quad (156)$$

$$\left(\frac{de'}{dt} \right)^P = \frac{de'}{dt} \Big|_{S_{(t,A,e')} \rightarrow S_{(t,A,e')}^P}^{\kappa'_t \rightarrow \kappa'_t{}^P}, \quad (157)$$

$$\left(\frac{d\varpi'}{dt} \right)^P = \frac{d\omega'}{dt} \Big|_{S_{(t,A,\omega')} \rightarrow S_{(t,A,\varpi')}^P}^{\kappa'_t \rightarrow \kappa'_t{}^P}. \quad (158)$$

9 Averaging Procedure

Until now, all the equations of motion describe how the secondary orbit changes throughout every revolution or flyby of every body in the system. Sometimes, however, the secondary is far away enough from some of the massive bodies such that their gravitational influence produces small oscillations of the secondary orbit. These oscillations may be averaged over any or all of the orbits to yield a net change in orbital elements. Averaged quantities are particularly useful to determine the long-term (often referred to as *secular*) evolution of a dynamical system, and provide fundamental insights that may be lost in the detail of the full, unaveraged equations.

I perform averaging only for $N = 3$ body systems, but consider every combination of averaging for these systems. For example, if the secondary is close to the primary but far from the tertiary, then one could average over only the secondary orbit or both orbits. The usefulness of either approach is dependent on the timescale for change sought, and the details of the system studied.

I denote the orbit average of an arbitrary variable β with a hat or a tilde such that averaging over the secondary and tertiary respectively is expressed as

$$\widehat{\frac{d\beta}{dt}} \equiv \frac{n}{2\pi} \int_0^{2\pi} \frac{d\beta}{dt} \frac{dt}{df} df, \quad \widetilde{\frac{d\beta}{dt}} \equiv \frac{n_t}{2\pi} \int_0^{2\pi} \frac{d\beta}{dt} \frac{dt}{df_t} df_t \quad (159)$$

and similarly for β' , with

$$\frac{dt}{df} = \frac{(1 - e^2)^{3/2}}{n(1 + e \cos f)^2}, \quad \frac{dt}{df_t} = \frac{(1 - e_t^2)^{3/2}}{n_t(1 + e_t \cos f_t)^2}. \quad (160)$$

The averaging procedure requires me to perform integrals which are difficult, if not impossible, to solve analytically with the equations of motion in their full generality. Therefore, I must make an approximation. So I utilize the power series representations generated from equation (82) and assume that either $\alpha_{sf} \ll 1$ or $\alpha_{sc} \ll 1$. My auxiliary variables are already written in forms to isolate f and f_t , facilitating the computation.

I use the algebraic manipulation software package *Mathematica* to perform the averaging, but must do so on a term-by-term basis. I find that the most expeditious procedure is to precompute individual integrals symbolically where the integrand is a function of $(1 + e \cos f)^q$ or $(1 + e' \cos f')^q$, where q is an integer. The result is in terms of hypergeometric functions of q . These symbolic solutions can then be used when computing coefficients to different orders. The integrals are also most easily computed when all powers of $\sin f$, $\cos f$, $\sin f'$, and $\cos f'$ are broken down into single powers through multiple-angle formulae.

I report final results to selected orders of powers of distances or semimajor axes depending on the length of the expressions. Often, double averaging yields much simpler formula than single averaging. However, the singly-averaged formulas may be important depending on the timescales considered. I compute averages for both the general equations of motion in the rotated frame and the equations of motion when all three bodies are coplanar.

10 Averaging when secondary is closer than tertiary

Here the relevant distance ratio is α_{sc} and the orbital elements are unprimed, meaning that they are measured with respect to the primary.

10.1 Averaging over secondary orbit only

The tertiary's orbit here may be eccentric, parabolic or hyperbolic. In the latter two cases, computing the resulting change in the secondary's orbital parameters may be particularly useful if the impulse approximation is not applicable.

10.1.1 Nonplanar equations

$$\widehat{\left(\frac{da}{dt}\right)}_{sc} = 0 \quad (161)$$

$$\widehat{\left(\frac{de}{dt}\right)}_{sc} = \left(\frac{1}{r_t^3}\right) \frac{15Gm_t e \sqrt{1 - e^2}}{2n} D_{t,1} D_{t,2} + \mathcal{O}\left(\frac{Gm_t}{n} \frac{a}{r_t^4}\right), \quad (162)$$

$$\begin{aligned}
\widehat{\left(\frac{di}{dt}\right)}_{\text{sc}} &= \left(\frac{1}{r_t^3}\right) \frac{3Gm_t \sin i \sin \chi_t}{2n\sqrt{1-e^2}} \\
&\times [D_{t,1} (1-e^2) \sin \omega + D_{t,2} (1+4e^2) \cos \omega] \\
&+ \mathcal{O}\left(\frac{Gm_t}{n} \frac{a}{r_t^4}\right), \tag{163}
\end{aligned}$$

$$\begin{aligned}
\widehat{\left(\frac{d\Omega}{dt}\right)}_{\text{sc}} &= \left(\frac{1}{r_t^3}\right) \frac{3Gm_t \sin \chi_t}{2n\sqrt{1-e^2}} \\
&\times [-D_{t,1} (1-e^2) \cos \omega + D_{t,2} (1+4e^2) \sin \omega] \\
&+ \mathcal{O}\left(\frac{Gm_t}{n} \frac{a}{r_t^4}\right), \tag{164}
\end{aligned}$$

$$\begin{aligned}
\widehat{\left(\frac{d\omega}{dt}\right)}_{\text{sc}} &= \left(\frac{1}{r_t^3}\right) \frac{3Gm_t}{2n\sqrt{1-e^2}} [5D_{t,2}^2 - 1 + e^2 - D_{t,1}^2 (1-e^2) \\
&- D_{t,2} (1+4e^2) \cos \omega \cos \chi_t \\
&+ D_{t,1} (1-e^2) \cos i \cos \omega \sin \chi_t] + \mathcal{O}\left(\frac{Gm_t}{n} \frac{a}{r_t^4}\right). \tag{165}
\end{aligned}$$

As a check on equations (161-165), I consider the expressions for the motion of a Martian satellite by Gurfil, Lainey & Efroimsky (2007). Those authors produce similar singly-averaged expressions, but for a satellite (secondary) of Mars (primary), which orbits the Sun (tertiary). The leading order terms in my equations (161-165) correctly reduce to their equations (38a-38e) under their assumption that the Martian orbit around the Sun is circular. Also, under this assumption, their $\tilde{\Omega}$ (their equation 41) is equivalent to my χ_t (my equation 44).

10.1.2 Coplanar equations

$$\widehat{\left(\frac{da}{dt}\right)}_{\text{sc}}^{\text{P}} = 0 \tag{166}$$

$$\begin{aligned}
\widehat{\left(\frac{de}{dt}\right)}_{\text{sc}}^{\text{P}} &= \left(\frac{1}{r_t^3}\right) \frac{15Gm_t e \sqrt{1-e^2}}{2n} D_{t,1}^{\text{P}} D_{t,2}^{\text{P}} \\
&+ \left(\frac{a}{r_t^4}\right) \frac{15Gm_t \sqrt{1-e^2}}{16n} D_{t,1}^{\text{P}} \\
&\times [4 + 3e^2 - 5 (D_{t,1}^{\text{P}})^2 (1-e^2) - 5 (D_{t,2}^{\text{P}})^2 (1+6e^2)] \\
&+ \mathcal{O}\left(\frac{Gm_t}{n} \frac{a^2}{r_t^5}\right), \tag{167}
\end{aligned}$$

$$\widehat{\left(\frac{d\varpi}{dt}\right)}_{\text{sc}}^{\text{P}} = -\left(\frac{1}{r_t^3}\right) \frac{3Gm_t \sqrt{1-e^2}}{2n} [1 + (D_{t,1}^{\text{P}})^2 - 4 (D_{t,2}^{\text{P}})^2]$$

$$\begin{aligned}
& + \left(\frac{a}{r_t^4} \right) \frac{15Gm_t \sqrt{1-e^2}}{16en} D_{t,2}^P \\
& \times \left[4 + 9e^2 - 5 (D_{t,1}^P)^2 (1 - 3e^2) - 5 (D_{t,2}^P)^2 (1 + 4e^2) \right] \\
& + \mathcal{O} \left(\frac{Gm_t}{n} \frac{a^2}{r_t^5} \right). \tag{168}
\end{aligned}$$

Although the averaged semimajor axis remains fixed, the eccentricity does not. If the eccentricity varies enough, then the secondary may collide with the primary or escape the system during a single orbit or flyby of the tertiary.

10.2 Averaging over both orbits

These equations should be used when one seeks the very long term evolution (over many tertiary orbits) of the system.

10.2.1 Nonplanar equations

Here I assume that the tertiary is on a bound (elliptical) orbit. Then

$$\widetilde{\left(\frac{da}{dt} \right)}_{\text{sc}} = 0, \tag{169}$$

$$\begin{aligned}
\widetilde{\left(\frac{de}{dt} \right)}_{\text{sc}} &= \left(\frac{1}{a_t^3} \right) \frac{15Gm_t e \sqrt{1-e^2} \sin 2\omega \sin^2 i}{8n (1-e_t^2)^{3/2}} \\
&- \left(\frac{a}{a_t^4} \right) \frac{15Gm_t e_t \sqrt{1-e^2}}{512n (1-e_t^2)^{5/2}} \\
&\times \left\{ \cos(\Omega - \varpi_t) (210e^2 \sin^2 i \sin(3\omega) \right. \\
&+ (3e^2 + 4) (5 \cos(2i) + 3) \sin \omega) \\
&+ 2 \sin(\Omega - \varpi_t) \cos i \cos \omega [7 (30e^2 \sin^2 i \cos(2\omega) - 9e^2 - 2) \\
&+ 15 (5e^2 + 2) \cos(2i)] \left. \right\} + \mathcal{O} \left(\frac{Gm_t}{n} \frac{a^2}{a_t^5} \right), \tag{170}
\end{aligned}$$

$$\begin{aligned}
\widetilde{\left(\frac{di}{dt} \right)}_{\text{sc}} &= - \left(\frac{1}{a_t^3} \right) \frac{15Gm_t e^2 \sin 2\omega \sin 2i}{16n \sqrt{1-e^2} (1-e_t^2)^{3/2}} \\
&+ \left(\frac{a}{a_t^4} \right) \frac{15Gm_t e e_t \sin i}{256n \sqrt{1-e^2} (1-e_t^2)^{5/2}} \\
&\times \left\{ 20 \cos(\Omega - \varpi_t) \cos i \sin \omega (7e^2 \cos(2\omega) + 5e^2 + 2) \right. \\
&- \sin(\Omega - \varpi_t) \cos \omega [- 35e^2 (3 \cos(2i) + 1) \cos(2\omega)
\end{aligned}$$

$$+ 15 (5e^2 + 2) \cos(2i) + 37e^2 + 26] \Big\} + \mathcal{O} \left(\frac{Gm_t}{n} \frac{a^2}{a_t^5} \right), \quad (171)$$

$$\begin{aligned} \widetilde{\left(\frac{d\Omega}{dt} \right)}_{\text{sc}} &= \left(\frac{1}{a_t^3} \right) \frac{3Gm_t \cos i (-2 - 3e^2 + 5e^2 \cos 2\omega)}{8n\sqrt{1-e^2} (1-e_t^2)^{3/2}} \\ &+ \left(\frac{a}{a_t^4} \right) \frac{15Gm_t e e_t}{256n\sqrt{1-e^2} (1-e_t^2)^{5/2}} \\ &\times \left\{ 20 \cos(\Omega - \varpi_t) \cos i \cos \omega (-7e^2 \cos(2\omega) + 5e^2 + 2) \right. \\ &- \sin(\Omega - \varpi_t) \sin \omega [-35e^2(3 \cos(2i) + 1) \cos(2\omega) \\ &+ 15(e^2 + 6) \cos(2i) + 17e^2 + 46] \Big\} + \mathcal{O} \left(\frac{Gm_t}{n} \frac{a^2}{a_t^5} \right), \quad (172) \end{aligned}$$

$$\begin{aligned} \widetilde{\left(\frac{d\omega}{dt} \right)}_{\text{sc}} &= \left(\frac{1}{a_t^3} \right) \frac{3Gm_t [2(1-e^2) - 5 \sin^2 \omega (\sin^2 i - e^2)]}{4n\sqrt{1-e^2} (1-e_t^2)^{3/2}} \\ &+ \left(\frac{a}{a_t^4} \right) \frac{15Gm_t e_t}{1024en\sqrt{1-e^2} (1-e_t^2)^{5/2}} \left\{ -4 \cos(\Omega - \varpi_t) \cos \omega \right. \\ &\times [89e^4 + 35e^2 \cos(2\omega) ((e^2 - 3) \cos(2i) - 5e^2 + 3) \\ &- 25e^2 + 5(-5e^4 + 17e^2 + 2) \cos(2i) + 6] \\ &- 2 \sin(\Omega - \varpi_t) \sin \omega \\ &\times [\cos i (8e^4 + 70e^2 \cos(2\omega) (4e^2 + 3 \cos(2i) - 3) - 237e^2 - 2) \\ &- 15(5e^2 + 2) \cos(3i)] \Big\} + \mathcal{O} \left(\frac{Gm_t}{n} \frac{a^2}{a_t^5} \right). \quad (173) \end{aligned}$$

Equations (170)-(173) are particularly important because the leading order term in each equation is the foundation of Lidov-Kozai theory (Lidov, 1961; Kozai, 1962), which was originally derived in the test particle limit. These terms are equivalent to equations (9.34) of Valtonen & Karttunen (2006) when their reduced mass of the primary and secondary is equal to unity, as they do not consider the test particle limit. Gurfil, Lainey & Efroimsky (2007) do consider the test particle limit, and their equations (42a-42e) match the leading order terms in my equations (170)-(173) in their limit of a circular tertiary orbit ($e_t = 0$).

I also included the next, (a/a_t^4) terms (often refereed to as the *octupole* terms) in full because of the interest they have recently attracted in the astronomical community. In particular, they can quantitatively affect classic Lidov-Kozai dynamics, a fact previously missed because of the premature elimination of the nodes in a Hamiltonian derivation (see the summary in Naoz et al., 2013). Here, these terms are derived without appealing to Delaunay variables, and explicitly demonstrate how the evolution becomes dependent on the longitude of ascending node. Note that all of these (a/a_t^4) terms vanish when the tertiary is on a circular orbit.

10.2.2 Coplanar equations

The planar versions of these terms are

$$\widetilde{\left(\frac{da}{dt}\right)}_{\text{sc}}^{\text{P}} = 0, \quad (174)$$

$$\begin{aligned} \widetilde{\left(\frac{de}{dt}\right)}_{\text{sc}}^{\text{P}} &= 0 \times \left(\frac{1}{a_t^3}\right) - \left(\frac{a}{a_t^4}\right) \frac{15Gm_t e_t \sqrt{1-e^2} (4+3e^2)}{64n (1-e_t^2)^{5/2}} \sin(\varpi - \varpi_t) \\ &\quad + \left(\frac{a^2}{a_t^5}\right) \frac{315Gm_t e_t^2 \sqrt{1-e^2} (2+e^2)}{256n (1-e_t^2)^{7/2}} \sin[2(\varpi - \varpi_t)] \\ &\quad + \mathcal{O}\left(\frac{Gm_t}{n} \frac{a^3}{a_t^6}\right), \end{aligned} \quad (175)$$

$$\begin{aligned} \widetilde{\left(\frac{d\varpi}{dt}\right)}_{\text{sc}}^{\text{P}} &= \left(\frac{1}{a_t^3}\right) \frac{3Gm_t \sqrt{1-e^2}}{4n (1-e_t^2)^{3/2}} \\ &\quad - \left(\frac{a}{a_t^4}\right) \frac{15Gm_t e_t \sqrt{1-e^2} (4+9e^2)}{64en (1-e_t^2)^{5/2}} \cos(\varpi - \varpi_t) \\ &\quad + \left(\frac{a^2}{a_t^5}\right) \frac{45Gm_t \sqrt{1-e^2}}{256n (1-e_t^2)^{7/2}} [(4+3e^2)(2+3e_t^2) \\ &\quad + 14(1+e^2)e_t^2 \cos[2(\varpi - \varpi_t)]] + \mathcal{O}\left(\frac{Gm_t}{n} \frac{a^3}{a_t^6}\right). \end{aligned} \quad (176)$$

Equations (174)-(176) impart important information.

- The leading order term ($1/a_t^3$) for the eccentricity variation vanishes, but that same term for the longitude of pericentre variation does not vanish. Therefore, to leading order, the longitude of pericentre evolution of the secondary can be solved for exactly. Consequently, $\widetilde{\varpi}_{\text{sc}}^{\text{P}} \propto t(1-e_t)^{-3/2}$, illustrating that the precession rate is faster for highly eccentric tertiaries.
- The eccentricity variation appears to vanish at all orders when the tertiary's orbit is circular, whereas for the variation of the longitude of pericentre, only every other term vanishes in this limit.
- The only term that is independent of both ϖ and ϖ_t is the leading term for the variation of the longitude of pericentre.
- Although all eccentricity variation terms appear to vanish when $\varpi - \varpi_t = 0$, this effect is instantaneous because the longitude of pericentre of the secondary is always precessing.

11 Averaging when tertiary is closer than secondary

Now I consider the far secondary case, where the tertiary-primary orbit is much smaller than the secondary-primary orbit. Assume both orbits are bounded orbits (osculating ellipses). Here, the primed semimajor axis always does vary after averaging, unlike in the close secondary case.

11.1 Averaging over tertiary orbit only

The leading order nonzero terms are long and I will not write them, but importantly I will indicate their leading nonzero order. Note that the order of the expansion becomes a function of the masses of the primary and tertiary. Every other of these terms vanishes in the special case of equal-mass binaries. Also, unlike in Section 11, in this subsection the evolution of f' must be averaged as well.

11.1.1 Nonplanar equations

$$\widetilde{\left(\frac{da'}{dt}\right)}_{\text{sf}} = \mathcal{O}\left(\frac{G}{n'} \frac{a_t^2}{r'^4} \frac{m_t m_p}{m_t + m_p}\right), \quad (177)$$

$$\widetilde{\left(\frac{de'}{dt}\right)}_{\text{sf}} = \mathcal{O}\left(\frac{G}{n'} \frac{a_t^2}{a'^2 r'^3} \frac{m_t m_p}{m_t + m_p}\right), \quad (178)$$

$$\widetilde{\left(\frac{di'}{dt}\right)}_{\text{sf}} = \mathcal{O}\left(\frac{G}{n'} \frac{a_t^2}{a'^2 r'^3} \frac{m_t m_p}{m_t + m_p}\right), \quad (179)$$

$$\widetilde{\left(\frac{d\Omega'}{dt}\right)}_{\text{sf}} = \mathcal{O}\left(\frac{G}{n'} \frac{a_t^2}{a'^2 r'^3} \frac{m_t m_p}{m_t + m_p}\right), \quad (180)$$

$$\widetilde{\left(\frac{d\omega'}{dt}\right)}_{\text{sf}} = \mathcal{O}\left(\frac{G}{n'} \frac{a_t^2}{a'^2 r'^3} \frac{m_t m_p}{m_t + m_p}\right), \quad (181)$$

$$\widetilde{\left(\frac{df'}{dt}\right)}_{\text{sf}} = \frac{n' (1 + e' \cos f')^2}{(1 - e'^2)^{3/2}} - \widetilde{\left(\frac{d\omega'}{dt}\right)}_{\text{sf}} - \cos i' \widetilde{\left(\frac{d\Omega'}{dt}\right)}_{\text{sf}}. \quad (182)$$

11.1.2 Coplanar equations

$$\widetilde{\left(\frac{da'}{dt}\right)}_{\text{sf}}^{\text{P}} = \mathcal{O}\left(\frac{G}{n'} \frac{a_t^2}{r'^4} \frac{m_t m_p}{m_t + m_p}\right), \quad (183)$$

$$\widetilde{\left(\frac{de'}{dt}\right)}_{\text{sf}}^{\text{P}} = \mathcal{O}\left(\frac{G}{n'} \frac{a_t^2}{a'^2 r'^3} \frac{m_t m_p}{m_t + m_p}\right), \quad (184)$$

$$\left(\widetilde{\frac{d\varpi'}{dt}}\right)_{\text{sf}}^{\text{P}} = \mathcal{O}\left(\frac{G}{n'} \frac{a_t^2}{a'^2 r'^3} \frac{m_t m_p}{m_t + m_p}\right), \quad (185)$$

$$\left(\widetilde{\frac{df'}{dt}}\right)_{\text{sf}}^{\text{P}} = \frac{n' (1 + e' \cos f')^2}{(1 - e'^2)^{3/2}} - \left(\widetilde{\frac{d\varpi'}{dt}}\right)_{\text{sf}}^{\text{P}}. \quad (186)$$

As shown, the change in semimajor axis is not zero, a marked difference from the previous averaged equations.

11.2 Averaging over both orbits

One may consider this case, at least to leading order, as a *reverse Lidov-Kozai* situation, where the perturber is internal.

11.2.1 Nonplanar equations

$$\left(\widetilde{\frac{da'}{dt}}\right)_{\text{sf}} = 0, \quad (187)$$

$$\begin{aligned} \left(\widetilde{\frac{de'}{dt}}\right)_{\text{sf}} &= 0 \times \left(\frac{a_t^2}{a'^5} \frac{m_t m_p}{m_t + m_p}\right) \\ &+ \mathcal{O}\left(\frac{G}{n'} \frac{a_t^3}{a'^6} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right), \end{aligned} \quad (188)$$

$$\begin{aligned} \left(\widetilde{\frac{di'}{dt}}\right)_{\text{sf}} &= \left(\frac{a_t^2}{a'^5} \frac{m_t m_p}{m_t + m_p}\right) \frac{15 G e_t^2 \sin i' \sin (2\Omega' - 2\varpi_t)}{8 n' (1 - e'^2)^2} \\ &+ \mathcal{O}\left(\frac{G}{n'} \frac{a_t^3}{a'^6} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right), \end{aligned} \quad (189)$$

$$\begin{aligned} \left(\widetilde{\frac{d\Omega'}{dt}}\right)_{\text{sf}} &= \left(\frac{a_t^2}{a'^5} \frac{m_t m_p}{m_t + m_p}\right) \frac{3 G \cos i' (-2 - 3e_t^2 + 5e_t^2 \cos (2\Omega' - 2\varpi_t))}{8 n' (1 - e'^2)^2} \\ &+ \mathcal{O}\left(\frac{G}{n'} \frac{a_t^3}{a'^6} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right), \end{aligned} \quad (190)$$

$$\begin{aligned} \left(\widetilde{\frac{d\omega'}{dt}}\right)_{\text{sf}} &= - \left(\frac{a_t^2}{a'^5} \frac{m_t m_p}{m_t + m_p}\right) \left(\frac{3 G}{32 n' (1 - e'^2)^2}\right) \\ &\times \left\{ 5e_t^2 (5 \cos (2i') - 1) \cos [(2\Omega' - 2\varpi_t)] - (2 + 3e_t^2) (3 + 5 \cos (2i')) \right\} \\ &+ \mathcal{O}\left(\frac{G}{n'} \frac{a_t^3}{a'^6} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right) \end{aligned} \quad (191)$$

The leading order term for the semimajor axis and eccentricity evolution vanishes (equations 187-188), but not for the evolution of the inclination, longitude of ascending node, nor the argument of pericentre. This striking observation allows me to consider obtaining a complete solution or stationary solution to these equations to leading order. Such solutions would also hold if the next order is included for equal-mass binaries, which would cause those terms to vanish.

Also striking is that to leading order, none of the equations are dependent on ω' . Therefore, the problem reduces to two variables and two equations (equations 189-190). I cannot find a complete solution, but at least one stationary solution does exist, when the secondary is on a polar orbit and the longitude of ascending node or longitude of descending node is equal to ϖ_t . When this configuration occurs, the argument of pericentre will still precess while the inclination and longitude of ascending node will remain static. The argument of pericentre will also become stationary at the critical value $e_{t,\text{crit}} = \sqrt{1/6} \approx 0.41$.

Equations (188)-(191) are in fact similar to their unprimed Lidov-Kozai counterparts (equations 170-173). The largest difference is that the variation in e' vanishes to leading order. The other variables have similar dependencies and forms except that notably all the leading-order primed variable terms are dependent on the secondary's longitude of ascending node.

11.2.2 Coplanar equations

$$\widehat{\left(\frac{da'}{dt}\right)}_{\text{sf}}^{\text{P}} = 0, \quad (192)$$

$$\begin{aligned} \widehat{\left(\frac{de'}{dt}\right)}_{\text{sf}}^{\text{P}} &= 0 \times \left(\frac{a_t^2}{a'^5}\right) \left(\frac{m_t m_p}{m_t + m_p}\right) \\ &\quad - \left(\frac{a_t^3}{a'^6} \frac{m_t m_p (m_p - m_t)}{(m_t + m_p)^2}\right) \frac{15G e_t (4 + 3e_t^2) \sin(\varpi' - \varpi_t)}{64n (1 - e'^2)^2} \\ &\quad + \mathcal{O}\left(\frac{G}{n'} \frac{a_t^4}{a'^7} \frac{m_t m_p (m_p^2 + m_t^2 - m_p m_t)}{(m_t + m_p)^3}\right), \end{aligned} \quad (193)$$

$$\begin{aligned} \widehat{\left(\frac{d\varpi'}{dt}\right)}_{\text{sf}}^{\text{P}} &= - \left(\frac{a_t^2}{a'^5}\right) \left(\frac{m_t m_p}{m_t + m_p}\right) \left[\frac{3G (2 + 3e_t^2)}{8n' (1 - e'^2)^2} \right] \\ &\quad - \left(\frac{a_t^3}{a'^6}\right) \left(\frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right) \\ &\quad \times \left[\frac{15G e_t (4 + 3e_t^2) (1 + 4e'^2) \cos(\varpi - \varpi_t)}{64n' e' (1 - e'^2)^3} \right] \end{aligned}$$

$$+ \mathcal{O} \left(\frac{G}{n'} \frac{a_t^4}{a'^7} \frac{m_t m_p (m_p^2 + m_t^2 - m_p m_t)}{(m_t + m_p)^3} \right) \quad (194)$$

As in the unprimed case, the evolution of the argument of pericentre dominates the change in the secondary's orbit. The change in eccentricity is a higher-order effect.

12 The Circular Circumbinary Case

Stellar systems containing tight binaries are common. These binaries usually have tidally circularized, which means that their current orbit is a circle. The behaviour of any particles or planets external to the binary is described by the primed element evolution equations in the special case of $e_t = 0$. This section presents this special case because of its importance and because the resulting simplification to the equations is significant, providing new insights into the motion.

12.1 Averaging over the binary companion (tertiary) only

Unlike in subsection 11.1, here the leading-order term of each equation is compact enough to present and potentially be useful.

12.1.1 Nonplanar equations

$$\begin{aligned} \left(\widetilde{\frac{da'}{dt}} \right) \Big|_{\text{sf}} \Big|_{e_t=0} &= - \left(\frac{a_t^2}{r'^4} \right) \left(\frac{m_t m_p}{m_t + m_p} \right) \left(\frac{3G}{2n' (1 - e'^2)^{1/2}} \right) \\ &\times \left\{ [e' \sin f' (5D'_{t,3}{}^2 + 5D'_{t,4}{}^2 - 2) \right. \\ &\quad \left. + 2D'_{t,3} D'_{t,5} - 2D'_{t,4} D'_{t,6}] \right\} \\ &+ \mathcal{O} \left(\frac{G}{n'} \frac{a_t^3}{r'^5} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2} \right), \end{aligned} \quad (195)$$

$$\begin{aligned} \left(\widetilde{\frac{de'}{dt}} \right) \Big|_{\text{sf}} \Big|_{e_t=0} &= \left(\frac{a_t^2}{a'^2 r'^3} \right) \left(\frac{m_t m_p}{m_t + m_p} \right) \left(\frac{3G}{8n' (1 - e'^2)^{1/2}} \right) \\ &\times \left\{ \sin f' (-5D'_{t,3}{}^2 - 5D'_{t,4}{}^2 + 2) (e' \cos f' + 1) \right. \\ &\quad \left. - D'_{t,3} D'_{t,7} + D'_{t,4} D'_{t,8} + e' \sin (2f') \right\} \\ &+ \mathcal{O} \left(\frac{G}{n'} \frac{a_t^3}{a'^2 r'^4} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2} \right), \end{aligned} \quad (196)$$

$$\begin{aligned}
\left(\widetilde{\frac{di'}{dt}}\right)\Big|_{\text{sf}|_{e_t=0}} &= -\left(\frac{a_t^2}{a'^2 r'^3}\right)\left(\frac{m_t m_p}{m_t + m_p}\right)\left(\frac{3G}{2n'(1-e'^2)^{1/2}}\right) \\
&\times \sin i' \cos(f' + \omega') \\
&\times \left\{D'_{t,4} \cos \Omega' - D'_{t,3} \sin \Omega'\right\} \\
&+ \mathcal{O}\left(\frac{G}{n'} \frac{a_t^3}{a'^2 r'^4} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right), \tag{197}
\end{aligned}$$

$$\begin{aligned}
\left(\widetilde{\frac{d\Omega'}{dt}}\right)\Big|_{\text{sf}|_{e_t=0}} &= -\left(\frac{a_t^2}{a'^2 r'^3}\right)\left(\frac{m_t m_p}{m_t + m_p}\right)\left(\frac{3G}{2n'(1-e'^2)^{1/2}}\right) \\
&\times \sin(f' + \omega') \\
&\times \left\{D'_{t,4} \cos \Omega' - D'_{t,3} \sin \Omega'\right\} \\
&+ \mathcal{O}\left(\frac{G}{n'} \frac{a_t^3}{a'^2 r'^4} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right), \tag{198}
\end{aligned}$$

$$\begin{aligned}
\left(\widetilde{\frac{d\omega'}{dt}}\right)\Big|_{\text{sf}|_{e_t=0}} &= \left(\frac{a_t^2}{a'^2 r'^3}\right)\left(\frac{m_t m_p}{m_t + m_p}\right)\left(\frac{3G}{8e'n'(1-e'^2)^{1/2}}\right) \\
&\times \left\{\left(5D_{t,3}'^2 + 5D_{t,4}'^2 - 2\right)(e' \cos(2f') + 2 \cos f')\right. \\
&\quad \left.+ 5e'D_{t,3}'^2 + 2D_{t,3}'D_{t,9}' - 2D_{t,4}'D_{t,10}' - 2e'\right\} \\
&+ \mathcal{O}\left(\frac{G}{n'} \frac{a_t^3}{a'^2 r'^4} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right), \tag{199}
\end{aligned}$$

$$\left(\widetilde{\frac{df'}{dt}}\right)\Big|_{\text{sf}|_{e_t=0}} = \frac{n'(1+e' \cos f')^2}{(1-e'^2)^{3/2}} - \left(\widetilde{\frac{d\omega'}{dt}}\right)\Big|_{\text{sf}|_{e_t=0}} - \cos i' \left(\widetilde{\frac{d\Omega'}{dt}}\right)\Big|_{\text{sf}|_{e_t=0}}. \tag{200}$$

In equations (195)-(200), the value of ϖ_t in the auxiliary D variables should be set to zero. The reason goes back to equations (30)-(32); if the tertiary has a forever circular orbit, x_t and y_t are parameterized by f_t alone. In this case, $f_t = \Pi_t = n_t t$, where Π_t represents the tertiary's mean anomaly.

12.1.2 Coplanar equations

$$\begin{aligned}
\left(\widetilde{\frac{da'}{dt}}\right)^{\text{P}}\Big|_{\text{sf}|_{e_t=0}} &= -\left(\frac{a_t^2}{r'^4}\right)\left(\frac{m_t m_p}{m_t + m_p}\right)\left(\frac{3G}{2n'(1-e'^2)^{1/2}}\right) \\
&\times \left\{e' \sin f' \left(5 \left[D_{t,3}^{\text{P}}\right]^2 + 5 \left[D_{t,4}^{\text{P}}\right]^2 - 2\right)\right\}
\end{aligned}$$

$$\begin{aligned}
& + 2D'_{t,3}D'_{t,5} - 2D'_{t,4}D'_{t,6}] \Big\} \\
& + \mathcal{O}\left(\frac{G}{n'} \frac{a_t^3}{r'^5} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right), \tag{201}
\end{aligned}$$

$$\begin{aligned}
\left(\widetilde{\frac{de'}{dt}}\right)_{\text{sf}}^{\text{P}} \Big|_{e_t=0} &= \left(\frac{a_t^2}{a'^2 r'^3}\right) \left(\frac{m_t m_p}{m_t + m_p}\right) \left(\frac{3G}{8n' (1 - e'^2)^{1/2}}\right) \\
&\times \left\{ \sin f' \left(-5 [D'_{t,3}]^2 - 5 [D'_{t,4}]^2 + 2 \right) (e' \cos f' + 1) \right. \\
&\quad \left. - D'_{t,3}D'_{t,7} + D'_{t,4}D'_{t,8} + e' \sin(2f') \right\} \\
&+ \mathcal{O}\left(\frac{G}{n'} \frac{a_t^3}{a'^2 r'^4} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right), \tag{202}
\end{aligned}$$

$$\begin{aligned}
\left(\widetilde{\frac{d\varpi'}{dt}}\right)_{\text{sf}}^{\text{P}} \Big|_{e_t=0} &= - \left(\frac{a_t^2}{a'^2 r'^3}\right) \left(\frac{m_t m_p}{m_t + m_p}\right) \left(\frac{3G}{8n' e' (1 - e'^2)^{1/2}}\right) \\
&\times \left\{ -2D'_{t,11}D'_{t,4} + 2D'_{t,12}D'_{t,3} \right. \\
&\quad + \left(-5 [D'_{t,3}]^2 - 5 [D'_{t,4}]^2 + 2 \right) (e' \cos(2f') + 2 \cos f') \\
&\quad \left. - 5e' \left([D'_{t,3}]^2 + [D'_{t,4}]^2 \right) + 2e' \right\} \\
&+ \mathcal{O}\left(\frac{G}{n'} \frac{a_t^3}{a'^2 r'^4} \frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right) \tag{203}
\end{aligned}$$

$$\left(\widetilde{\frac{df'}{dt}}\right)_{\text{sf}}^{\text{P}} \Big|_{e_t=0} = \frac{n' (1 + e' \cos f')^2}{(1 - e'^2)^{3/2}} - \left(\widetilde{\frac{d\varpi'}{dt}}\right)_{\text{sf}}^{\text{P}} \Big|_{e_t \rightarrow 0} \tag{204}$$

12.2 Averaging over both orbits

The following equations highlight the importance of the higher-order functional dependencies of the binary masses. For some perspective about how steeply the magnitude of the mass function changes, successive orders for equal-mass binaries yield $(\frac{m_p}{2}, 0, \frac{m_p}{8}, 0, \frac{m_p}{32})$ and for $m_p = 2m_t$ yield

$$\left(\frac{m_p}{3}, \frac{m_p}{9}, \frac{m_p}{9}, \frac{m_p}{27}, \frac{11m_p}{273}\right).$$

12.2.1 Nonplanar equations

$$\widehat{\left(\frac{da'}{dt}\right)}_{\text{sf}} \Big|_{e_t=0} = 0, \quad (205)$$

$$\begin{aligned} \widehat{\left(\frac{de'}{dt}\right)}_{\text{sf}} \Big|_{e_t=0} &= 0 \times \left(\frac{a_t^2}{a'^5}\right) \left(\frac{m_t m_p}{m_t + m_p}\right) \\ &+ 0 \times \left(\frac{a_t^3}{a'^6}\right) \left(\frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right) \\ &+ \left(\frac{a_t^4}{a'^7}\right) \left(\frac{m_t m_p (m_p^2 + m_t^2 - m_p m_t)}{(m_t + m_p)^3}\right) \\ &\times \frac{45 G e' \sin^2 i' \sin(2\omega') (5 + 7 \cos(2i'))}{512 n' (1 - e'^2)^3} \\ &+ \mathcal{O} \left(\frac{G a_t^5 m_t m_p (m_p^3 + m_t^3 - m_p^2 m_t - m_p m_t^2)}{n' a'^8 (m_t + m_p)^4} \right), \quad (206) \end{aligned}$$

$$\begin{aligned} \widehat{\left(\frac{di'}{dt}\right)}_{\text{sf}} \Big|_{e_t \rightarrow 0} &= 0 \times \left(\frac{a_t^2}{a'^5}\right) \left(\frac{m_t m_p}{m_t + m_p}\right) \\ &+ 0 \times \left(\frac{a_t^3}{a'^6}\right) \left(\frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right) \\ &- \left(\frac{a_t^4}{a'^7}\right) \left(\frac{m_t m_p (m_p^2 + m_t^2 - m_p m_t)}{(m_t + m_p)^3}\right) \\ &\times \frac{45 G e'^2 \sin 2i' \sin(2\omega') (5 + 7 \cos(2i'))}{1024 n' (1 - e'^2)^4} \\ &+ \mathcal{O} \left(\frac{G a_t^5 m_t m_p (m_p^3 + m_t^3 - m_p^2 m_t - m_p m_t^2)}{n' a'^8 (m_t + m_p)^4} \right), \quad (207) \end{aligned}$$

$$\begin{aligned} \widehat{\left(\frac{d\Omega'}{dt}\right)}_{\text{sf}} \Big|_{e_t=0} &= - \left(\frac{a_t^2}{a'^5}\right) \left(\frac{m_t m_p}{m_t + m_p}\right) \frac{3G \cos i'}{4n' (1 - e'^2)^2} \\ &+ 0 \times \left(\frac{a_t^3}{a'^6}\right) \left(\frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2}\right) \\ &+ \left(\frac{a_t^4}{a'^7}\right) \left(\frac{m_t m_p (m_p^2 + m_t^2 - m_p m_t)}{(m_t + m_p)^3}\right) \\ &\times \frac{G}{1024 n' (1 - e'^2)^4} \left[90 e'^2 \cos(2\omega') (5 \cos i' + 7 \cos(3i')) \right] \end{aligned}$$

$$\begin{aligned}
& - 45 (2 + 3e'^2) (9 \cos i' + 7 \cos(3i')) \Big] \\
& + \mathcal{O} \left(\frac{G}{n'} \frac{a_t^5}{a'^8} \frac{m_t m_p (m_p^3 + m_t^3 - m_p^2 m_t - m_p m_t^2)}{(m_t + m_p)^4} \right), \quad (208)
\end{aligned}$$

$$\begin{aligned}
\widehat{\left(\frac{d\omega'}{dt} \right)}_{\text{sf}} \Big|_{e_t=0} &= \left(\frac{a_t^2}{a'^5} \right) \left(\frac{m_t m_p}{m_t + m_p} \right) \frac{3G (3 + 5 \cos(2i'))}{16n' (1 - e'^2)^2} \\
&+ 0 \times \left(\frac{a_t^3}{a'^6} \right) \left(\frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2} \right) \\
&+ \left(\frac{a_t^4}{a'^7} \right) \left(\frac{m_t m_p (m_p^2 + m_t^2 - m_p m_t)}{(m_t + m_p)^3} \right) \\
&\times \frac{45G}{8129n' (1 - e'^2)^4} \\
&\times \left[4 \cos 2i' (52 + 63e'^2 + \cos(2\omega') (4 - 14e'^2)) \right. \\
&- 7 \cos 4i' [2 \cos(2\omega') (2 + 9e'^2) - 27e'^2 - 28] \\
&\left. + 2 \cos(2\omega') (6 - 5e'^2) + 27 (4 + 5e'^2) \right] \\
&+ \mathcal{O} \left(\frac{G}{n'} \frac{a_t^5}{a'^8} \frac{m_t m_p (m_p^3 + m_t^3 - m_p^2 m_t - m_p m_t^2)}{(m_t + m_p)^4} \right). \quad (209)
\end{aligned}$$

Notably, the leading-order nonzero terms for the change in Ω' and ω' are (a_t^2/a'^5) , whereas for e' and i' the terms are of order (a_t^4/a'^7) . Hence, the orbital change is dominated by orientation variations as opposed to stretching or warping. The two leading order terms are immediately solvable because neither are functions of Ω' nor ω' . Hence, the precession of both the pericentre and node proceed linearly with time to an excellent approximation. The precession rate is dependent on a' , e' and i' . This rate vanishes for Ω' only for polar orbits, and for ω' only at a critical value $i'_{\text{crit}} = (1/2) \cos^{-1}[-3/5] \approx 63.4^\circ$.

12.2.2 Coplanar equations

The nonzero leading-order term in equation (206) vanishes when $i' = 0$, suggesting that the planar equations in circular circumbinary systems afford even greater simplification.

$$\widehat{\left(\frac{da'}{dt} \right)}_{\text{sf}}^{\text{P}} \Big|_{e_t=0} = 0 \quad (210)$$

$$\left. \widehat{\left(\frac{de'}{dt} \right)}_{\text{sf}}^{\text{P}} \right|_{e_t=0} = 0 \quad (211)$$

$$\begin{aligned} \left. \widehat{(\varpi')}_{\text{sf}}^{\text{P}} \right|_{e_t=0} &= t \left(\frac{a_t^2}{a'^5} \right) \left(\frac{m_t m_p}{m_t + m_p} \right) \left[\frac{3G}{4n' (1 - e'^2)^2} \right] \\ &+ 0 \times \left(\frac{a_t^3}{a'^6} \right) \left(\frac{m_t m_p (m_t - m_p)}{(m_t + m_p)^2} \right) \\ &+ t \left(\frac{a_t^4}{a'^7} \right) \left[\frac{m_t m_p (m_p^2 + m_t^2 - m_p m_t)}{(m_t + m_p)^3} \right] \\ &\times \left[\frac{45G (4 + 3e'^2)}{128n' (1 - e'^2)^4} \right] \\ &+ 0 \times \left(\frac{a_t^5}{a'^8} \right) \left(\frac{m_t m_p (m_p^3 + m_t^3 - m_p^2 m_t - m_p m_t^2)}{(m_t + m_p)^4} \right) \\ &+ t \left(\frac{a_t^6}{a'^9} \right) \left[\frac{m_t m_p (m_p^4 + m_t^4 - m_p^3 m_t - m_t^3 m_p + m_t^2 m_p^2)}{(m_t + m_p)^5} \right] \\ &\times \left[\frac{525G [8 + 5e'^2 (4 + e'^2)]}{2048n' (1 - e'^2)^6} \right] + \dots \end{aligned} \quad (212)$$

Equation (212) is not written as a differential equation because a full solution is available out to an order of at least (a_t^6/a'^9) . This solution, which is linear with time, exists because the evolution of the argument of pericentre is independent of itself. The dependence, which exists in the nonplanar version (equation 209), vanishes in the coplanar limit. Hence, I achieve a complete solution to the first three nonzero orders. Computing additional terms becomes challenging, and may not be particularly useful. What is useful is that equation (212) satisfies an arbitrarily high value of e' .

Nevertheless, the equation provides a good opportunity to link this formalism to established dynamical theory. Now I demonstrate how equations (211-212) reduce to coplanar Laplace-Lagrange secular theory, a popular treatment of which is described in Chapter 7 of Murray & Dermott (1999). The theory enables one to obtain approximate doubly-averaged solutions of the equations of motion in the three-body problem in limited situations. One limitation is that the eccentricities of the bodies must be small. The classic treatment expands the eccentricities out to second order. In contrast, equation (212) satisfies any value of e' .

Rather than compare the results to the classic theory, I compare the results to the more expansive fourth-order Laplace-Lagrange theory (Veras & Armitage 2007). Equation (8) of that paper provides the necessary disturbing function, to be used in conjunction with the Lagrange's equations of motion in their equations (10c) and (10d), which importantly differ from the standard

reduced versions given in equations (7.16) of Murray & Dermott (1999). Retaining disturbing function eccentricity terms to fourth order and expanding equations (10c) and (10d) from Veras & Armitage (2007) about small eccentricity yields exactly the same coefficients from equation (212) of this paper when expanded about small eccentricity, to at least the first two nonzero orders in semimajor axis ratio and eccentricity. The mass functions naturally differ because of the different setups. Also, note that the secondary eccentricity evolution similarly vanishes in coplanar Laplace-Lagrange theory when $e_t = 0$.

13 Resonances

Mean motion resonances may occur in the restricted three-body problem just as in the unrestricted problem. Notably, none of the equations presented so far require a distinction to be made; the equations satisfy both resonant and non-resonant behaviour. Now I place the formalism in the context of resonances.

In a three body system, a mean motion resonance between the secondary and tertiary is helped defined by the following time-dependent angle

$$\phi = q_{\lambda_{\text{out}}} \lambda_{\text{out}} + q_{\lambda_{\text{in}}} \lambda_{\text{in}} + q_{\varpi_{\text{out}}} \varpi_{\text{out}} + q_{\varpi_{\text{in}}} \varpi_{\text{in}} + q_{\Omega_{\text{out}}} \Omega_{\text{out}} + q_{\Omega_{\text{in}}} \Omega_{\text{in}} \quad (213)$$

where the mean longitude $\lambda \equiv \varpi + \Pi$, the mean anomaly $\Pi \equiv E - e \sin E$, E is the eccentric anomaly, and the constants q add to zero. The subscripts “in” and “out” refer to the chosen ordering of the secondary and tertiary with respect to distance from the primary.

The time evolution of ϕ determines whether the secondary and tertiary are in a particular single-argument resonance defined by the q values. Typical analytical treatments of obtaining $\dot{\phi}$ utilize a disturbing function that is truncated in orders of eccentricity, where a dot denotes a time derivative. Here, I show how an explicit relation for $\dot{\phi}$ with arbitrarily high eccentricities is obtained using the equations in this paper.

If the (zero-mass) secondary is the inner body, then its orbital elements are measured with respect to the primary. If the secondary is the outer body, then instead the primed elements should be used, which are measured with respect to the centre of mass of the primary and tertiary. Without loss of generality for this exercise, assume the former case. The tertiary’s orbit will never change, and hence $\dot{\varpi}_{\text{out}} = \dot{\Omega}_{\text{out}} = 0$. Also, $\dot{\lambda}_{\text{out}} = \dot{\Pi}_{\text{out}} = n_{\text{out}}$ is known.

For the secondary, I obtain

$$\dot{\lambda}_{\text{in}} = \dot{\omega}_{\text{in}} + \dot{\Omega}_{\text{in}} + \dot{E}_{\text{in}} (1 - e_{\text{in}} \cos E_{\text{in}}) - \dot{e} \sin E_{\text{in}}. \quad (214)$$

Also, from equation (12) of Veras & Evans (2013)

$$\dot{E}_{\text{in}} = \frac{\sqrt{1 - e_{\text{in}}^2}}{1 + e_{\text{in}} \cos f_{\text{in}}} \dot{f}_{\text{in}} - \frac{\sin f_{\text{in}}}{\sqrt{1 - e_{\text{in}}^2} (1 + e_{\text{in}} \cos f_{\text{in}})} \dot{e}_{\text{in}}. \quad (215)$$

Using equation (215) along with the standard relations

$$\cos E_{\text{in}} = \frac{e_{\text{in}} + \cos f_{\text{in}}}{1 + e_{\text{in}} \cos f_{\text{in}}}, \quad \sin E_{\text{in}} = \frac{\sin f_{\text{in}} \sqrt{1 - e_{\text{in}}^2}}{1 + e_{\text{in}} \cos f_{\text{in}}} \quad (216)$$

gives

$$\dot{\lambda}_{\text{in}} = \dot{\omega}_{\text{in}} + \dot{\Omega}_{\text{in}} + \frac{(1 - e_{\text{in}}^2)^{3/2}}{(1 + e_{\text{in}} \cos f_{\text{in}})^2} \dot{f}_{\text{in}} - \frac{\sin f_{\text{in}} \sqrt{1 - e_{\text{in}}^2} (2 + e_{\text{in}} \cos f_{\text{in}})}{(1 + e_{\text{in}} \cos f_{\text{in}})^2} \dot{e}_{\text{in}}. \quad (217)$$

Finally, I use equation (20) to make the substitution for \dot{f}_{in} , yielding

$$\begin{aligned} \dot{\phi} = & q_{\lambda_{\text{out}}} n_{\text{out}} + q_{\lambda_{\text{in}}} \frac{n_{\text{in}} (1 + e_{\text{in}} \cos f_{\text{in}})^2}{(1 - e_{\text{in}}^2)^{3/2}} \\ & - q_{\lambda_{\text{in}}} \frac{\sin f_{\text{in}} \sqrt{1 - e_{\text{in}}^2} (2 + e_{\text{in}} \cos f_{\text{in}})}{(1 + e_{\text{in}} \cos f_{\text{in}})^2} \dot{e}_{\text{in}} \\ & + \left[q_{\lambda_{\text{in}}} \left(1 - \frac{(1 - e_{\text{in}}^2)^{3/2}}{(1 + e_{\text{in}} \cos f_{\text{in}})^2} \right) + q_{\varpi_{\text{in}}} \right] \dot{\varpi}_{\text{in}} \\ & + \left[q_{\lambda_{\text{in}}} \left(1 - \frac{\cos i_{\text{in}} (1 - e_{\text{in}}^2)^{3/2}}{(1 + e_{\text{in}} \cos f_{\text{in}})^2} \right) + q_{\varpi_{\text{in}}} + q_{\Omega_{\text{in}}} \right] \dot{\Omega}_{\text{in}}. \end{aligned} \quad (218)$$

Now one can use whichever set of equations for $\{\dot{e}_{\text{in}}, \dot{\omega}_{\text{in}}, \dot{\Omega}_{\text{in}}\}$ from this paper which are appropriate to the system being studied. Consequently, the time evolution of the resonant angle is expressed entirely in terms of orbital elements with no time derivatives on the RHS.

Subsequently, regardless of whether one expands α about 0 to reduce the resulting equation, note there is no need to expand separately about e_{in} nor i_{in} about 0 also. Hence, one can model resonant angles with high values of the eccentricity and inclination.

14 Summary

I have derived the equations of motion in the general restricted N -body problem as functions of $(a, e, i, \Omega, \omega, f)$ only (equations 4-20). I then expressed these relations with respect to an orbital plane that is fixed in space (equations 49-54), which represents a practical application in many contexts (e.g. a mass-hierarchic N -body problem, or the Solar System's ecliptic). Modeling hyperbolic instead of elliptical orbits requires only a change in the definition of p_j (equation 33). These equations, along with their partially coplanar (equations 76-81) and fully coplanar (equations 102-110) versions, are not subject to any averagings nor expansions nor assumptions about small forces. This formulation may facilitate the study of mean motion resonances (equation 218).

Orbital elements measured with respect to the centres of mass of a particular set of bodies require just a translation from the primary-centric case (equations 138 and 225), although alternatively the explicit equations of motion may be used (equations 144-149).

I applied some of the above equations to three-body systems, and presented singly- and doubly-averaged expressions for nonplanar and planar configurations for the primary-centric (Section 10), general barycentric (Section 11), and circular circumbinary barycentric (Section 12) cases. This procedure, can, for example, yield Lidov-Kozai terms to a desired order (equations 170-173). The averaged equations reveal the dominant drivers of orbital changes, and identify which elements remain stationary over long timescales when measured with respect to the barycentre of the binary. I also find exact solutions to leading order (equations 194 and 208-209) and to several orders (equation 212) for the precession of the pericentre and node in circular circumbinary systems with an eccentric external body. A solution to leading order also exists (equation 176) for coplanar wide binary systems.

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A Appendix: Barycentric equations for an arbitrary number of bodies

Here I derive the equations of motion in orbital elements for the secondary with respect to the barycentre of 3 or more massive bodies. This section is an extension of Section 8, which treats the two-massive-body case. Examples of physical situations in which the equations here may be applied include a distant terrestrial planet orbiting in a circumbinary planetary system already containing a Hot Jupiter, or a free-floating distant comet or planet being

gravitationally captured and kept by a trinary or quaternary stellar system. The procedure entails achieving an equation similar in form to equations (1)-(3), as in equation (137).

Assume \mathbf{r} is the vector from the primary to the (zero-mass) secondary, and \mathbf{s} is the vector from the secondary to the centre of mass of all the massive bodies. I need to develop the key equation of motion in terms of \mathbf{s} . Assume the total number of bodies⁶ in the system is $N \geq 3$, and for ease of summation indexing, here let the primary be the first body (denoted with a subscript 1) and the secondary be the N th body. Further assume that the vector from the primary to the $u = 2 \dots (N-1)$ th body is \mathbf{r}_u , and the vector from the centre of mass of the system to the $j = 1 \dots (N-1)$ th body is denoted by \mathbf{s}_j .

The equation of motion for \mathbf{r} is

$$\frac{d^2 \mathbf{r}}{dt^2} = -\frac{Gm_1 \mathbf{r}}{r^3} + \sum_{u=2}^{N-1} \left[Gm_u \frac{(\mathbf{r}_u - \mathbf{r})}{|\mathbf{r}_u - \mathbf{r}|^3} - Gm_u \frac{\mathbf{r}_u}{r_u^3} \right]. \quad (219)$$

I need to express equation (219) in terms of \mathbf{s} and \mathbf{r}_u for $u = 2 \dots (N-1)$. In order to make a substitution for \mathbf{r} on the RHS, consider the relation for the centre of mass of the system

$$\sum_{u=1}^{N-1} m_u \mathbf{s}_u = 0. \quad (220)$$

Also, by simultaneously solving $(2N-4)$ vector triangle relations, along with equation (220), I obtain

$$\mathbf{r} = \frac{\sum_{u=2}^{N-1} m_u \mathbf{r}_u}{\sum_{j=1}^{N-1} m_j} - \mathbf{s}. \quad (221)$$

Equation (221) can be used on the RHS of equation (219). To obtain an expression for $d^2 \mathbf{r}/dt^2$, I use both equation (221) and the following standard equations of motion for $u = 2 \dots (N-1)$ massive bodies

$$\frac{d^2 \mathbf{r}_u}{dt^2} + \frac{G(m_1 + m_u) \mathbf{r}_u}{r_u^3} = \sum_{\substack{j=2 \\ j \neq u}}^{N-1} Gm_j \left(\frac{(\mathbf{r}_j - \mathbf{r}_u)}{|\mathbf{r}_j - \mathbf{r}_u|^3} - \frac{\mathbf{r}_j}{r_j^3} \right) \quad (222)$$

to obtain

$$\frac{d^2 \mathbf{r}}{dt^2} = -\frac{d^2 \mathbf{s}}{dt^2} - \sum_{u=2}^{N-1} Gm_u \frac{\mathbf{r}_u}{r_u^3}. \quad (223)$$

Finally, equations (220) and (223) give the desired form of equation (219) as

$$\begin{aligned} \frac{d^2 \mathbf{s}}{dt^2} = & - \left(\sum_{u=1}^{N-1} Gm_u \right) \frac{\mathbf{s}}{s^3} + Gm_1 \frac{\frac{\sum_{u=2}^{N-1} m_u \mathbf{r}_u}{\sum_{j=1}^{N-1} m_j} - \mathbf{s}}{\left| \frac{\sum_{u=2}^{N-1} m_u \mathbf{r}_u}{\sum_{j=1}^{N-1} m_j} - \mathbf{s} \right|^3} \\ & - \left[\sum_{u=2}^{N-1} Gm_u \frac{\frac{[(\sum_{j=1}^{N-1} m_j) - m_u] \mathbf{r}_u - [(\sum_{j=2}^{N-1} m_j \mathbf{r}_j) - m_u \mathbf{r}_u]}{\sum_{w=1}^{N-1} m_w} + \mathbf{s}}{\left| \frac{[(\sum_{j=1}^{N-1} m_j) - m_u] \mathbf{r}_u - [(\sum_{j=2}^{N-1} m_j \mathbf{r}_j) - m_u \mathbf{r}_u]}{\sum_{w=1}^{N-1} m_w} + \mathbf{s} \right|^3} \right] \\ & + \left(\sum_{u=1}^{N-1} Gm_u \right) \frac{\mathbf{s}}{s^3}. \end{aligned} \quad (224)$$

⁶ The barycentre need not be with respect to all massive bodies of the system, usually just the massive bodies which are interior to the secondary.

Let the orbital elements measured with respect to this centre of mass be denoted with a asterisk. The equations of motion for an arbitrary variable β^* are finally

$$\begin{aligned}
\frac{d\beta^*}{dt} = & \frac{m_1}{m_t} \left(\frac{d\beta}{dt} \right)_{t,B} \bigg|_{\mathbf{r}_t \rightarrow \left[\sum_{j=1}^{N-1} m_j \right]^{-1} \left[\sum_{u=2}^{N-1} m_u \mathbf{r}_u \right]}^{(a,e,i,\Omega,\omega,f) \rightarrow (a^*,e^*,i^*,\Omega^*,\omega^*,f^*)} \\
& + \left[\sum_{u=2}^{N-1} \left(\frac{d\beta}{dt} \right)_{u,B} \bigg|_{\mathbf{r}_u \rightarrow - \left[\sum_{w=1}^{N-1} m_w \right]^{-1} \left\{ \left[\left(\sum_{j=1}^{N-1} m_j \right) - m_u \right] \mathbf{r}_u - \left[\left(\sum_{j=2}^{N-1} m_j \mathbf{r}_j \right) - m_u \mathbf{r}_u \right] \right\}}^{(a,e,i,\Omega,\omega,f) \rightarrow (a^*,e^*,i^*,\Omega^*,\omega^*,f^*)} \right] \\
& - \frac{\left(\sum_{j=1}^{N-1} m_j \right)}{m_t} \left(\frac{d\beta}{dt} \right)_{t,B} \bigg|_{\mathbf{r}_t \rightarrow 0}^{(a,e,i,\Omega,\omega,f) \rightarrow (a^*,e^*,i^*,\Omega^*,\omega^*,f^*)}. \tag{225}
\end{aligned}$$

In equation (225), the tertiary-specific terms may be substituted for any body which is not the primary nor secondary. Because the scalings in the first and second terms involve linear combinations of distances, the resulting expressions for the orbital element evolution will involve multiple distance ratios.